Optimal Topology Control and Power Allocation for Minimum Energy Consumption in Consensus Networks

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***Abstract*—Consensus algorithms have generated a lot of interest due to their ability to compute globally relevant statistics by only exploiting local communications among sensors. However, when implemented over wireless sensor networks, the inherent iterative nature of consensus algorithms may cause a large energy consumption. Hence, to make consensus algorithms really appealing in sensor networks, it is necessary to minimize the energy necessary to reach a consensus, within a given accuracy. We propose a method to optimize the network topology and the power allocation over each active link in order to minimize the energy consumption. We consider two network models: a deterministic model, where the nodes are located arbitrarily but their positions are known, and a random model, where the network topology is modeled as a random geometric graph (RGG). In the first case, we show how to convert the topology optimization problem, which is inherently combinatorial, into a parametric convex problem, solvable with efficient algorithms. In the second case, we optimize the power transmitted by each node, exploiting the asymptotic distributions of the eigenvalues of the adjacency matrix of an RGG. We further show that the optimal power can be found as the solution of a convex problem. The theoretical findings are corroborated with extensive simulation results.**

***Index Terms*—Consensus algorithms, minimum energy consumption, random graphs, sensor networks, topology control.**

I. Introduction

AVERAGE consensus algorithms have received considerable attention in recent years because of their ability to enable globally optimal decisions using only local exchange of information among nearby nodes [1]–[3]. The price paid for this simplicity and the underlying decentralized philosophy is that consensus algorithms are inherently iterative. As a consequence, the implementation of consensus algorithms over a wireless sensor network (WSN) requires an iterated exchange of data among the nodes, which might cause an excessive energy consumption. This must be contrasted with a centralized strategy where there is a sink node that, after collecting all the observations from the sensors (perhaps over multiple hops), is virtually able to compute the desired statistic in a single shot. Hence, to make consensus algorithms practically appealing in a sensor network context, it is necessary to minimize the energy consumption necessary to reach consensus. Clearly, the network topology plays a fundamental role in determining the convergence rate [4]. It is well known that, as the network connectivity increases, so does the rate of convergence. However, having a densely connected network requires a high power consumption to guarantee reliable direct links between many nodes. In principle, having a fully connected network is equivalent to having as many sink nodes as sensors, so that the convergence time of fully connected networks is minimum. However, the power consumption necessary to maintain a fully connected network is also maximum. On the other hand, a minimally connected network entails low power consumption to maintain a few links, but, at the same time, it requires a large convergence time. Since what really matters in a WSN is the overall power spent to achieve consensus, this paper addresses the problem of finding the optimal network topology that minimizes this overall power consumption, taking into account convergence time and transmit power necessary to establish reliable links *jointly*. The search for the optimal topology is, *per se*, a combinatorial problem whose solution becomes prohibitive even for small scale networks. Nevertheless, we will present a relaxation technique that converts the problem into a convex problem, with minimum performance loss.

If the links among the nodes are symmetric, or, equivalently, if the graph describing the network topology is undirected, the convergence rate can be lower bounded through the so called *algebraic connectivity*, defined as the second smallest eigenvalue of the graph Laplacian [5]. For this reason, there have been works aimed at maximizing the algebraic connectivity of a given undirected graph by a suitable choice of the weights associated to each edge [6], [7]. Alternatively, in [8] it was shown how some network topologies, such as small world graphs, can greatly increase the convergence rate. In [9], the authors show that nonbipartite Ramanujan graphs constitute a class of topologies maximizing the convergence rate. In [10] it was shown how to add edges from a given set to a graph in order to maximize its algebraic connectivity. Other works, for example [11], proposed strategies to improve the convergence rate of gossip algorithms through geographic routing.

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However, in all these works, the focus is always on convergence time and there is no cost associated with the establishment of the graph topology. Conversely, in our work, since the graph represents a real network, we consider as the optimization metric the power consumption necessary to maintain reliable communication links among the nodes, taking into account the radio propagation model, multiplied by the number of iterations necessary to achieve consensus. From this perspective, enforcing a small world, scale-free, or Ramanujan graph topology may not be the best choice for a wireless network, whose topology should depend on the application at hand as well as on the propagation model.

In the wireless communication network context, there have been works on optimizing the network topology in order to minimize the power consumption necessary to guarantee connectivity, e.g., [12], [13]. These works concentrate on the communication task and do not make any specific reference to the running application. However, it is now well established, in the sensor network context, that, whenever possible, an efficient wireless sensor network design should take into account the specific goal of the sensor network [14]. For this reason, we focus on the achievement of consensus in a WSN. It is well known that one of the most crucial parameters in WSNs is energy consumption, because in many contexts it is hard to recharge the batteries or scavenge energy from the environment. For this reason, in most applications, minimizing energy consumption is more appropriate than minimizing convergence time (although, there are important applications where the latter could be more important). In [15], the minimum energy consumption problem was studied, assuming a common transmit power. As shown in [15], there typically is an optimum power that minimizes the energy necessary to achieve consensus within a prescribed accuracy.

In this work, we generalize the initial idea suggested in [15] and we propose a method for optimizing the network topology and the power allocation across every link in order to minimize the energy necessary to achieve consensus. We consider two classes of networks: a) deterministic topologies, with arbitrary, but known, node locations, and b) random geometries, with unknown node locations, modeled as random variables. In the deterministic case, we optimize both topology and power allocation. Differently from [6], we do not assume any prior topology, as the topology comes out as a result of the optimization. Topology optimization is, in general, a combinatorial problem and hence an NP-hard problem (recall that an undirected graph composed of n nodes may assume $2^{n(n-1)/2}$ topologies). To tackle this issue, we propose a relaxation technique that allows us to formulate topology optimization as a convex parametric problem. Then, we show that the effect of this relaxation on the performance is negligible.

In the random topology case, where the internode distances are unknown, we show how to optimize the single (common) transmit power, modeling the network topology as a random geometric graph, a model suitable for wireless networks. We provide both theoretical and simulation results, exploiting the theory of the eigenvalues of random geometric graphs.

The paper is organized as follows. In Section II, we briefly review the consensus algorithm. In Section III, we introduce our communication model and formulate the optimization problem. Section IV is devoted to topology optimization for arbitrary networks. In Section V, we start by providing a closed form expression, albeit approximate, for the algebraic connectivity of a random geometric graph. Then we use this expression to proceed with the topology optimization for random geometric graphs. The analytical findings are corroborated with extensive simulation results.

II. Brief Review of Consensus Algorithms

Let us consider a wireless network composed of n sensors. The network topology can be represented as an undirected graph $G = \{V, E\}$ where V denotes the set of n vertices (nodes) v_i and $E \subseteq V \times V$ is the set of bidirectional edges (links) $e_{ij} = e_{ji}$ connecting v_i and v_j. Furthermore, let $\boldsymbol{A}$ be the $n \times n$-dimensional symmetric *adjacency* matrix of the graph G, with elements $a_{ij} = 1$ if $e_{ij} \in E$ and $a_{ij} = 0$ otherwise. According to this notation and assuming no self-loops, i.e., $a_{ii} = 0$, $\forall i = 1, \ldots, n$, the degree of node v_i is defined as $\deg(v_i) = \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$. The *degree* matrix $\boldsymbol{D}$ is defined as the diagonal matrix whose diagonal entry is $d_{ii} = \deg(v_i)$. Let $\mathcal{N}_i$ denote the set of neighbors of node i, so that $|\mathcal{N}_i| = \deg(v_i)$[1]. The *Laplacian* matrix of the graph G is the $n \times n$ symmetric matrix $\boldsymbol{L} := \boldsymbol{D} - \boldsymbol{A}$, whose entries are

$$\ell_{ij} = \ell_{ji} = \begin{cases} \deg(v_i) & \text{if } j = i \\ -a_{ij} & \text{if } j \neq i \end{cases}. \tag{1}$$

Given a set of measurements $x_i(0)$, collected by node i, for $i = 1, \ldots, n$, the goal of the consensus algorithm is to allow every node to compute a globally optimal function of the measurements, say x^*, through a decentralized mechanism that does not require the presence of a sink node, but builds on the interaction among nearby nodes only. Let us consider, for simplicity, the case where x^* is the average of the measurements. In this case, reaching an average consensus can be seen as the minimization of the disagreement between the states x_i of the interacting nodes. One of the nice properties of the Laplacian is that the disagreement can be expressed as a quadratic form built on the Laplacian [16]

$$\begin{aligned} J(\boldsymbol{x}) &:= \frac{1}{4} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} (x_i - x_j)^2 \\ &= \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (x_i - x_j)^2 = \frac{1}{2} \boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x}. \end{aligned} \tag{2}$$

An important property of the Laplacian is that it has, by construction, a zero eigenvalue, whose multiplicity is equal to the number of connected components of a graph. Hence, a graph is connected if its zero eigenvalue has multiplicity one [5]. Furthermore, if the graph is connected, the eigenvector associated with the null eigenvalue is the vector $\mathbf{1}$, composed of all ones.

The minimization of the quadratic form in (2) can be achieved using a simple steepest descent technique. In continuous time, the minimum of (2) can be reached through the following updating rule [2]:

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L} \boldsymbol{x}(t) \tag{3}$$

[1]By $|\cdot|$ we denote the cardinality of the set.

initialized with $\boldsymbol{x}(0) := \boldsymbol{x}_0$[2]. By construction, the eigenvalues of $\boldsymbol{L}$ are all nonnegative. Hence, the convergence of (3) is guaranteed. In particular, if the graph is connected, the state vector $\boldsymbol{x}(t)$ in (3) converges to the projection of the initial state $\boldsymbol{x}_0$ onto the nullspace of $\boldsymbol{L}$, spanned by the vector $\mathbf{1}$, i.e., [2]

$$\boldsymbol{x}(t) \underset{t\to\infty}{\longrightarrow} \frac{1}{n}\,\mathbf{1}\,\mathbf{1}^T\boldsymbol{x}_0. \tag{4}$$

This corresponds to having all the nodes converging to a consensus on the average of the initial observations $x^* = \sum_{i=1}^n x_i(0)/n$. Moreover, the convergence rate is lower bounded by the slowest decaying mode, i.e., by the second smallest eigenvalue of $\boldsymbol{L}$, $\lambda_2(\boldsymbol{L})$, also known as the *algebraic connectivity* of the graph. More specifically, if the graph is connected, the dynamic system (3) converges to consensus exponentially [2], i.e., $\|\boldsymbol{x}(t) - x^*\mathbf{1}\| \leq \|\boldsymbol{x}(0) - x^*\mathbf{1}\|O(e^{-rt})$, with $r = \lambda_2(\boldsymbol{L})$. Defining the disagreement vector as $\Delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - x^*\mathbf{1}$, we can write $\|\Delta\boldsymbol{x}(t)\| \leq \|\Delta\boldsymbol{x}(0)\|O(e^{-rt})$. As a consequence, the convergence time T_c can be defined (see also [2]) as the time necessary for the slowest mode of the dynamical system (3) to be reduced by a factor $\gamma \ll 1$, i.e., the time for which $\|\Delta\boldsymbol{x}(T_c)\| \leq \gamma\|\Delta\boldsymbol{x}(0)\|$. Hence, we can set

$$T_c = -\frac{\log(\gamma)}{\lambda_2(\boldsymbol{L})}. \tag{5}$$

III. Optimization Criterion

It is well known that one of the most critical issues in WSN is energy consumption. Neglecting for simplicity the power spent for processing operations with respect to the power to be used to enable wireless communications, the overall power spent to reach consensus is the product between the sum of the power P_{tot} necessary to establish the communication links among the nodes and the number of iterations N_{it} necessary to achieve consensus. The exchange of information among the nodes is supposed to take place in the presence of a slotted system, with a medium access control (MAC) mechanism that prevents packet collisions. Denoting by T_s the duration of a time slot unit, the number of iterations is then approximately[3] $N_{\text{it}} = T_c/T_s$.

Introducing the power coefficients p_{ij}, $i \neq j$, denoting the power used by node i to transmit to node j with $p_{ij} = p_{ji}$, and the binary coefficients a_{ij} assessing the presence $(a_{ij} = 1)$ of a link between nodes i and j or not $(a_{ij} = 0)$, the power spent by the whole network in each iteration is then $P_{\text{tot}} = \sum_{i,j} a_{ij}p_{ij}$. Using (5), our goal is to minimize the following performance metric:

$$\mathcal{E} = P_{\text{tot}}N_{\text{it}} = K\frac{\sum_{i=1}^n\sum_{j=1}^n a_{ij}p_{ij}}{\lambda_2(\boldsymbol{L}(\boldsymbol{a}))} \tag{6}$$

[2]The discrete-time counterpart of (3) is [2]

$$\boldsymbol{x}[k+1] = \boldsymbol{x}[k] - \tilde{\epsilon}\boldsymbol{L}\boldsymbol{x}[k] := \boldsymbol{W}\boldsymbol{x}[k]$$

where $\tilde{\epsilon}$ is a parameter chosen so as to ensure that no eigenvalue of $\boldsymbol{W}$ is greater than one in modulus.

[3]We neglect rounding errors, which tend to vanish if the duration of the updating time slot is small with respect to the convergence time.

where K incorporates all irrelevant constants. This metric is proportional to the integral of power consumption with respect to time and then it represents an energy consumption. In (6), we have made explicit the dependence of the Laplacian $\boldsymbol{L}$ on the vector $\boldsymbol{a} = \boldsymbol{A}(:)$ containing all binary coefficients a_{ij}, since finding these coefficients is the goal of our optimization. More specifically, our goal is to find the set of active links, i.e., the nonzero coefficients a_{ij}, and the powers p_{ij} that minimize (6), under the constraint of maintaining network connectivity, which entails $\lambda_2(\boldsymbol{L}(\boldsymbol{a})) > 0$. The problem can then be formulated as follows:

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{p}} \quad & \frac{\sum_{i=1}^n\sum_{j=1}^n a_{ij}p_{ij}}{\lambda_2(\boldsymbol{L}(\boldsymbol{a}))} \\ \text{s.t.} \quad & \epsilon \leq \lambda_2(\boldsymbol{L}(\boldsymbol{a})) \quad [\mathbf{P.0}] \\ & a_{ij} \in \{0,1\} \\ & p_{ij} \geq 0 \quad \forall\, i,j = 1,\ldots,n \end{aligned} \tag{7}$$

where ϵ is an arbitrarily small positive constant used to prevent the algebraic connectivity from going to zero, which would correspond to a disconnected network and $\boldsymbol{p}$ is the vector with entries p_{ij}.

Since the topology coefficients are binary variables, $[\mathbf{P.0}]$ is a combinatorial problem, with complexity increasing with the size n of the network as $2^{n(n-1)/2}$. Hence, its solution, for medium/large scale networks is prohibitive. Our objective is to modify $[\mathbf{P.0}]$ in order to turn it into a convex problem, with negligible performance losses.

A first simplification comes from observing that the coefficients a_{ij} and p_{ij} are not independent of each other. Their dependence is indeed a consequence of the radio propagation model. In this work, given the complexity of the topology optimization problem, we assume a fairly simple communication model. We state that there is a link between nodes i and j, and then $a_{ij} = 1$, if the signal–to-noise ratio SNR_j at the receiver node j, when node i transmits, exceeds a minimum value $s_{\min}$, i.e., $\text{SNR}_j > s_{\min}$. If we denote by p_{Rj} the power received by node j when node i transmits, and by σ_n^2 the noise power, assumed for simplicity to be the same at each receiving node, we have $a_{ij} = 1$, if $p_{Rj} > s_{\min}\sigma_n^2 := p_{\min}$, or otherwise $a_{ij} = 0$. Assuming flat fading channel modeling, we use the following propagation model:

$$p_{Rj} = \frac{p_{ij}}{1 + (r_{ij}/r_0)^\eta} \tag{8}$$

where r_{ij} is the distance between nodes i and j, and η is the path loss exponent. The parameter r_0 plays the role of a scaling factor or reference distance, and typically corresponds to the so called Fraunhofer distance, such that, if $r_{ij} \gg r_0$, the receiver is in the transmit antenna far-field, where the received power is inversely proportional to r_{ij}^η; conversely, if $r_{ij} \ll r_0$, the receiver is in the transmit antenna near-field, where the received power is approximately equal to the transmitted one. The unity term in the denominator of (8) is used to avoid the unrealistic situation in which the received power could be greater than the transmitted one. Given the propagation model (8), the relation

between the power coefficients p_{ij} and the topology coefficients a_{ij} is then

$$a_{i\,j} = \begin{cases} 1 & \text{if } p_{ij} > p_{\min}\left[1+\left(\frac{r_{ij}}{r_0}\right)^{\eta}\right] \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

In the following sections we will show how to relax this relation in order to simplify the solution of the optimal topology control problem. We will consider two scenarios: a) a deterministic topology, with arbitrary node geometry, where the distances between the nodes are known, and b) a random topology, where the positions of the nodes are unknown and modeled as random variables. In case a), since the distances among the nodes are known, we optimize the power allocated to each link. This strategy is also instrumental in determining the topology, which is equivalent to finding the $n(n-1)/2$ entries a_{ij} of the adjacency matrix. Conversely, in the random topology case, since the distances are not known, we assume broadcast communications, and look for the optimization of the coverage radius of each node. The deterministic topology case will be the subject of Section IV, whereas the random case will be studied in Section V.

IV. Optimal Topology and Power Allocation for Arbitrary Networks

In the case where the distances between the nodes are known, the optimization criterion amounts to solving problem $[\mathbf{P.0}]$, which involves a combinatorial strategy that makes the problem numerically very hard to solve, especially for medium/large scale networks. The relation (9) reduces the set of unknowns to the set p_{ij}, but the problem still retains most of its difficulties. To make problem $[\mathbf{P.0}]$ to be numerically tractable, we introduce a first relaxation so that, instead of requiring a_{ij} to be binary, we assume a_{ij} to be a real variable belonging to the interval $[0,1]$. Under this assumption, problem $[\mathbf{P.0}]$ is not combinatorial anymore, but it is still a nontrivial nonlinear constrained problem. The first important contribution of this paper is to propose a relaxation technique that transforms the previous problem into a *convex* problem that can be solved with well established and efficient numerical tools. We achieve this goal by first introducing the following relationship between the coefficients a_{ij} and the distances r_{ij}:

$$a_{ij} = \frac{1}{1+(r_{ij}/r_{c_{ij}})^{\alpha}} \tag{10}$$

where α is a positive coefficient and $r_{c_{ij}}$ is the coverage radius, which depends on the transmit power. According to (10), a_{ij} is close to one when node j is within the coverage radius of node i, i.e., $r_{ij} \ll r_{c_{ij}}$, whereas a_{ij} is close to zero, when $r_{ij} \gg r_{c_{ij}}$. The switching from zero to one can be made steeper by increasing the value of α.

Given the propagation model (8), the coverage radius $r_{c_{ij}}$ is related to the power p_{ij} and the minimum power required for reliable communication $p_{\min}$, as follows:

$$r_{c_{ij}} = r_0\left(\frac{p_{ij}}{p_{\min}}-1\right)^{1/\eta}. \tag{11}$$

Plugging (11) in (10), the coefficients a_{ij} can be written explicitly in terms of the power coefficients as follows:

$$a_{ij} = a_{ij}(p_{ij}) = \frac{r_0^{\alpha}(p_{ij}-p_{\min})^{\alpha/\eta}}{r_0^{\alpha}(p_{ij}-p_{\min})^{\alpha/\eta} + r_{ij}^{\alpha}p_{\min}^{\alpha/\eta}}. \tag{12}$$

This relation can be also inverted to find the coefficients p_{ij} as a function of a_{ij}, as follows:

$$p_{ij} = q(a_{ij}) = p_{\min} + k_1\left(\frac{a_{ij}}{1-a_{ij}}\right)^{\eta/\alpha} \tag{13}$$

with $k_1 = p_{\min}\frac{r_{ij}^{\eta}}{r_0^{\eta}}$. Expression (12) becomes our relaxed version of (9) and it allows us to reduce the set of variables to the only power vector $\boldsymbol{p}$. Consequently, problem $[\mathbf{P.0}]$ can be relaxed into the following problem:

$$\begin{aligned} \min_{\boldsymbol{p}} \quad & \frac{\boldsymbol{p}^T\mathbf{1}}{\lambda_2(\boldsymbol{L}(\boldsymbol{p}))} \\ \text{s.t.} \quad & \epsilon \le \lambda_2(\boldsymbol{L}(\boldsymbol{p})) \quad [\mathbf{P.1}] \\ & p_{\min}\mathbf{1} \le \boldsymbol{p} \end{aligned} \tag{14}$$

where, thanks to (12), the Laplacian is now written explicitly in terms of the power coefficients p_{ij}. In principle, the last inequality in (14) makes any link feasible. But this does not imply that the final network will be fully connected, because some coefficients a_{ij} might turn out to be equal zero, implying that the link between node i and j is not active. The first important result, related to the solution of (14), is the following.

Theorem 1: Given the propagation model in (8), using the relations (12) between the topology coefficients a_{ij} and the power terms p_{ij}, problem $[\mathbf{P.1}]$ is a convex-concave fractional problem if $\eta \ge \alpha$.

Proof: Let us consider the objective function in $[\mathbf{P.1}]$. The numerator of $[\mathbf{P.1}]$ is clearly a convex function of $\boldsymbol{p}$. We only have to prove the concavity of the algebraic connectivity $\lambda_2(\boldsymbol{L}(\boldsymbol{p}))$ with respect to the transmit powers. As a first step, we prove that $a_{ij}(p_{ij})$ is a concave function of p_{ij}. Then, we use this to show that $\lambda_2(\,\cdot\,)$ is a concave function of $\boldsymbol{p}$.

(*i*) *$a_{ij}(p_{ij})$ is a concave function of p_{ij}*: We compute the second-order derivative of the function in (12) with respect to p_{ij}

$$\frac{d^2a_{ij}(p_{ij})}{d^2p_{ij}} = k_3[(\alpha-\eta)k_4 - k_2(\alpha+\eta)(p_{ij}-p_{\min})^{\alpha/\eta}] \tag{15}$$

where the constants k_2, k_3 and k_4 are given by

$$\begin{aligned} & k_2 = r_0^{\alpha},\ k_3 = \frac{\alpha k_4 k_2 (p_{ij}-p_{\min})^{\alpha/\eta-2}}{[k_4+k_2(p_{ij}-p_{\min})^{\alpha/\eta}]^3\,\eta^2}, \\ & k_4 = r_{ij}^{\alpha}p_{\min}^{\alpha/\eta}. \end{aligned}$$

Note that $k_2, k_4 > 0$; since $p_{ij} \ge p_{\min}\ \forall i,j$, we also have $k_3 \ge 0$. From (15), we see that, if $\alpha \le \eta$, the second-order derivative is always nonpositive and then $a_{ij}(p_{ij})$ is a concave function of p_{ij}.

(*ii*) *$\lambda_2(\,\cdot\,)$ is concave in $\boldsymbol{p}$*: Exploiting the properties of the Laplacian [16], we can write the quadratic form associated to $\boldsymbol{L}$ as in (2). From (12), we also note that each coefficient a_{ij}

depends only on the corresponding link power p_{ij}, and not on the other link powers. Given any pair of power vectors $\boldsymbol{p}^{(1)}$ and $\boldsymbol{p}^{(2)}$, let $\boldsymbol{p}=\beta\boldsymbol{p}^{(1)}+(1-\beta)\boldsymbol{p}^{(2)}$, with $0\leq\beta\leq 1$. Hence, we have

$$\begin{aligned}\boldsymbol{x}^T\boldsymbol{L}(\boldsymbol{a}(\boldsymbol{p}))\boldsymbol{x}&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(p_{ij})(x_i-x_j)^2\\&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\left(\beta p_{ij}^{(1)}+(1-\beta)p_{ij}^{(2)}\right)(x_i-x_j)^2\\&\geq\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\beta a_{ij}\left(p_{ij}^{(1)}\right)+(1-\beta)a_{ij}\left(p_{ij}^{(2)}\right)\right](x_i-x_j)^2\\&=\beta\boldsymbol{x}^T\boldsymbol{L}\left(\boldsymbol{a}\left(\boldsymbol{p}^{(1)}\right)\right)\boldsymbol{x}+(1-\beta)\boldsymbol{x}^T\boldsymbol{L}\left(\boldsymbol{a}\left(\boldsymbol{p}^{(2)}\right)\right)\boldsymbol{x}\end{aligned}\tag{16}$$

where the inequality follows from the concavity of $a_{ij}(p_{ij})$.

We further recall that the algebraic connectivity $\lambda_2(\boldsymbol{L})$ is the solution of the following positive semidefinite program (SDP) [5]:

$$\lambda_2(\boldsymbol{L})=\min_{\boldsymbol{x}\perp\mathbf{1},\|\boldsymbol{x}\|=1}\boldsymbol{x}^T\boldsymbol{L}\boldsymbol{x}.\tag{17}$$

Hence, using (16), we obtain

$$\begin{aligned}&\lambda_2\left(\boldsymbol{L}\left(\beta\boldsymbol{p}^{(1)}+(1-\beta)\boldsymbol{p}^{(2)}\right)\right)\\&\quad\geq\lambda_2\left(\beta\boldsymbol{L}\left(\boldsymbol{p}^{(1)}\right)+(1-\beta)\boldsymbol{L}\left(\boldsymbol{p}^{(2)}\right)\right)\\&\quad\geq\beta\lambda_2\left(\boldsymbol{L}\left(\boldsymbol{p}^{(1)}\right)\right)+(1-\beta)\lambda_2\left(\boldsymbol{L}\left(\boldsymbol{p}^{(2)}\right)\right)\end{aligned}$$

where the last equality follows from a majorization theorem for eigenvalues[4]. Thus, the algebraic connectivity is a concave function of the vector of transmit powers. Hence, (14) is a convex-concave fractional problem since it is the ratio of a convex and a concave function of $\boldsymbol{p}$. ■

Since (14) is a convex-concave fractional problem, we can use one of the methods that solve quasi-convex optimization problems, see, e.g., [18], [19]. For example, we can use the nonlinear parametric formulation proposed in [19]. To do so, we introduce the following function:

$$h(\mu)=\min\{\boldsymbol{p}^T\mathbf{1}-\mu\lambda_2(\boldsymbol{L}(\boldsymbol{p})):\boldsymbol{p}\in\Gamma\}$$

where μ is a real positive parameter and $\Gamma=\{\boldsymbol{p}:\boldsymbol{p}\geq\mathbf{1}p_{\min},\lambda_2(\boldsymbol{L}(\boldsymbol{p}))\geq\epsilon\}$. In order to find the solution of this problem we could use the following result, proved in [19] and [20].

Theorem 2: Let $f(x)$ and $g(x)$ be continuous real-valued functions $\forall x\in\Theta$, where Θ is a nonempty compact subset of $\mathbb{R}^n$ and $g(x)>0\ \forall x\in\Theta$. Then

$$\mu^*=\frac{f(x^*)}{g(x^*)}=\min\left\{\frac{f(x)}{g(x)}:x\in\Theta\right\}$$

with $x^*\in\Theta$, if, and only if

$$h^*=h(\mu^*,x^*)=\min\{f(x)-\mu^*g(x):x\in\Theta\}=0$$

where $h(\mu^*,x^*)$ means that for $\mu=\mu^*$ the minimum of $\{f(x)-\mu^*g(x):x\in\Theta\}$ is taken on at $x=x^*$.

Before applying Theorem 2, it is useful to further convert the convex-concave optimization problem $[\mathbf{P.1}]$ into the following parametric problem:

$$\begin{aligned}\min_{\boldsymbol{p}}\quad&\boldsymbol{p}^T\mathbf{1}-\mu\lambda_2(\boldsymbol{L}(\boldsymbol{p}))\\\text{s.t.}\quad&\epsilon\leq\lambda_2(\boldsymbol{L}(\boldsymbol{p}))\qquad[\mathbf{P.2}]\\&\mathbf{1}p_{\min}\leq\boldsymbol{p}.\end{aligned}\tag{18}$$

By Theorem 1, $\lambda_2(\boldsymbol{L}(\boldsymbol{p}))$ is a concave function of $\boldsymbol{p}$. Hence, the objective function in $[\mathbf{P.2}]$, as sum of convex functions, is a convex function. The constraint sets are convex. Then, problem $[\mathbf{P.2}]$ is a convex parametric problem, whose solution is a function of the parameter μ that controls the tradeoff between the global transmit power and the convergence time. Later on, we will show how to find the optimal μ.

Since problem $[\mathbf{P.2}]$ is convex, it can be solved using numerically efficient convex programming tools. However, before applying any convex tool, it is worth noticing that the feasible set in $[\mathbf{P.2}]$ is not compact. Hence, even if a solution exists, in principle, it could be unreachable in finite time. To overcome this potential drawback, we propose next an alternative formulation of $[\mathbf{P.2}]$: Instead of looking for the set of power coefficients p_{ij}, and then for the a_{ij}, using (13), we can reformulate $[\mathbf{P.2}]$ so as to look directly for the variables a_{ij}. Then the optimization problem in (18) can be rewritten in terms of vector $\boldsymbol{a}$, as follows:

$$\begin{aligned}\min_{\boldsymbol{a}}\quad&\phi(\boldsymbol{a})-\mu\lambda_2(\boldsymbol{L}(\boldsymbol{a}))\\\text{s.t.}\quad&\epsilon\leq\lambda_2(\boldsymbol{L}(\boldsymbol{a}))\qquad[\mathbf{P.3}]\\&\mathbf{0}\leq\boldsymbol{a}<\mathbf{1}\end{aligned}\tag{19}$$

where $\phi(\boldsymbol{a})=\sum_{i=1}^{n}\sum_{\substack{j=1\\i\neq j}}^{n}q(a_{ij})$. We verify next that this problem is still convex. To study the behavior of $\phi(\boldsymbol{a})$, we compute the second-order derivative of $q(a_{ij})$, obtaining

$$\frac{d^2q(a_{ij})}{da_{ij}^2}=k_1\frac{\eta}{\alpha}\left(\frac{a_{ij}}{1-a_{ij}}\right)^{\eta/\alpha-2}\frac{1}{(1-a_{ij})^4}\left(\frac{\eta}{\alpha}-1+2a_{ij}\right).$$

We infer that

$$\frac{d^2q(a_{ij})}{da_{ij}^2}\geq 0\Leftrightarrow\eta-\alpha+2\alpha a_{ij}\geq 0.$$

We note that, if $\eta\geq\alpha$, then $\frac{d^2q(a_{ij})}{da_{ij}^2}\geq 0$ for $0\leq a_{ij}<1$, so that $\phi(\boldsymbol{a})$, as a sum of convex functions, is convex[5]. Finally, the algebraic connectivity $\lambda_2(\boldsymbol{L}(\boldsymbol{a}))$ is a concave function of $\boldsymbol{a}$, as can be proved following the same steps as in Theorem

[4]Let $\boldsymbol{A}$, $\boldsymbol{B}$ be Hermitian matrices, and let $\boldsymbol{C}=\boldsymbol{A}+\boldsymbol{B}$. Let their eigenvalues be sorted in nondecreasing order. Then the vector of eigenvalues $\lambda(\boldsymbol{C})$ majorizes the vector $\lambda(\boldsymbol{A})+\lambda(\boldsymbol{B})$. Since $\lambda_1(\boldsymbol{L})=0$ for the graph Laplacian, it follows that $\lambda_2(\boldsymbol{A}+\boldsymbol{B})\geq\lambda_2(\boldsymbol{A})+\lambda_2(\boldsymbol{B})$. See [17, Theorem 4.3.27].

[5]Note that $\phi(\boldsymbol{a})$ is the sum of functions of the single variables a_{ij}. Hence the convexity of $\phi(\boldsymbol{a})$ can be studied looking at the convexity of the single functions $q(a_{ij})$.

1. Then, the optimization problem $[\mathbf{P.3}]$ is also a convex parametric problem, perfectly equivalent to the original problem in (18), since the change of variables $p_{ij} = q(a_{ij})$ in (13) ensures a one-to-one mapping $q : \mathbb{R} \rightarrow \mathbb{R}$, for $0 \leq a_{ij} < 1$, with image covering the problem domain in (18) (see [21, p. 130]). Hence, assuming $\mathbf{0} \leq \boldsymbol{a} \leq \mathbf{1} - \epsilon^{'}$, with $\epsilon^{'}$ an infinitesimal positive constant so that the feasible set of (19) $\Upsilon = \{\boldsymbol{a} : \mathbf{0} \leq \boldsymbol{a} \leq \mathbf{1} - \epsilon^{'}, \lambda_2(\boldsymbol{L}(\boldsymbol{a})) \geq \epsilon\}$ is a compact convex set in $\mathbb{R}^{n(n-1)/2}$, we ensure that the set of minima of (19) is nonempty and by the convexity of the problem we can deduce that all local minima are also global. Note that since the optimization problem in (19) is convex an optimal solution can be found via efficient numerical tools. Furthermore, using Dinkelbach's algorithm [19], based on Theorem 2, we are also able to find the optimal parameter μ in $[\mathbf{P.3}]$. More specifically, the Dinkelbach's algorithm, applied to our problem, proceeds through the following steps:

1. Set $i = 1$ and let $\boldsymbol{a}_i$ be a feasible point of Υ, with $\mu_i = \frac{\phi(\boldsymbol{a}_i)}{\lambda_2(\boldsymbol{L}(\boldsymbol{a}_i))}$;
2. Set $\mu = \mu_i$ and find $\boldsymbol{a}_{i+1} \in \Upsilon$ that solves the minimization problem in $[\mathbf{P.3}]$;
3. If $| h(\mu, \boldsymbol{a}_{i+1}) | = | \phi(\boldsymbol{a}_{i+1}) - \mu\lambda_2(\boldsymbol{L}(\boldsymbol{a}_{i+1})) | \leq \epsilon''$, with ϵ'' an arbitrarily small positive constant, stop and take $\boldsymbol{a}_{i+1}$ as the optimal link coefficient vector; otherwise, set $i = i+1$, $\mu_i = \frac{\phi(\boldsymbol{a}_i)}{\lambda_2(\boldsymbol{L}(\boldsymbol{a}_i))}$ and go to step 2.

Since the topology coefficients a_{ij} obtained in this way are real variables belonging to the interval $[0, 1]$, to obtain the network topology, it is necessary to quantize them to convert them into binary values, 1 or 0, indicating the presence or absence of a link. This quantization is achieved by comparing each a_{ij} with a threshold $a_{\rm th}$. Of course, the final topology will depend on the threshold value. Moreover, the thresholding operation will also affect the final result in terms of convergence time and energy consumption. It is then of interest to check how sensitive the final topology, as well as convergence time and energy consumption, are to the choice of $a_{\rm th}$. In the ensuing section, we present some numerical results to shed light on the resulting topologies and their dependence on the propagation model parameters.

A. Numerical Examples

Since our optimization procedure is based on a relaxation technique, the first important step is to evaluate the impact of the relaxation on the final topology and performance.

Example 1: Comparison Between Exhaustive Search and Relaxed Technique: We compare now the topology obtained as a solution of the relaxed problem $[\mathbf{P.3}]$ with the optimal graph obtained by solving directly problem $[\mathbf{P.0}]$ using an exhaustive search over all possible topologies. For complexity reasons, of course we can only perform this comparison for small scale networks. We consider networks of $n = 4$ and 6 nodes.

To provide results not conditioned to a specific geographic node deployment, we averaged the results over 100 statistically independent realizations of the nodes locations. In each iteration we compute the minimum energy $\mathcal{E}_{\rm op}$ reached using the optimal exhaustive search over all possible topologies, and the energy $\mathcal{E}_r$ corresponding to the network topology whose coefficients a_{ij} are obtained by solving problem $[\mathbf{P.3}]$ and thresholding the result with the threshold $a_{\rm th} = 2 \times 10^{-3}$. In Fig. 1, we report the average energies $\mathcal{E}_{\rm op}$ and $\mathcal{E}_r$ versus η. We can note from Fig. 1 that the loss in terms of optimal energy due to the relaxation of the original problem is negligible (the energy loss is less than 1.7 dB).

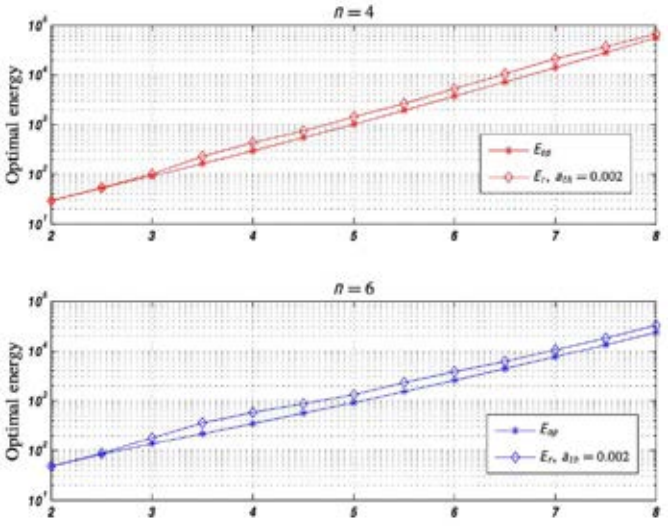

Fig. 1. Optimal average energy obtained with exhaustive search and average energy obtained by solving the relaxed problem $[\mathbf{P.3}]$ versus η, for different n values.

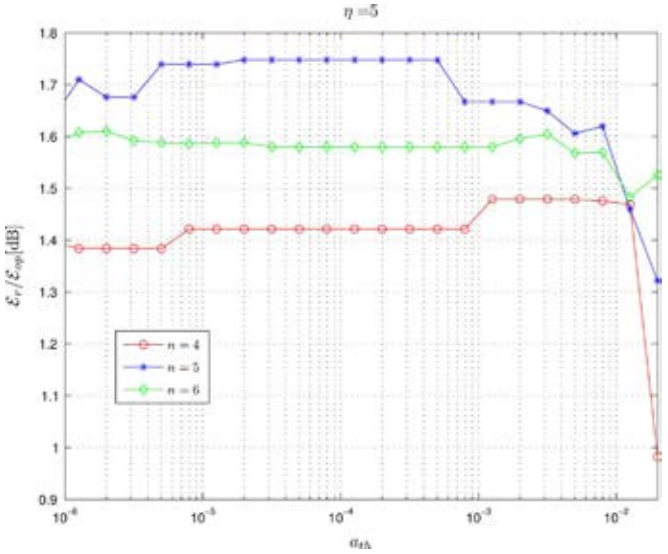

Fig. 2. Ratio between the optimal energy obtained with exhaustive search and the energy obtained by solving the proposed relaxed problem $[\mathbf{P.3}]$, versus $a_{\rm th}$, for different values of n.

Example 2: Impact of Thresholding Operation on Final Topology: Clearly, the selection of the threshold $a_{\rm th}$ plays a role in the identification of the final topology. To evaluate the impact of $a_{\rm th}$ on the final topology, in Fig. 2 we report the ratio $\frac{\mathcal{E}_r}{\mathcal{E}_{\rm op}}$ versus $a_{\rm th}$ for different numbers of nodes n and for $\eta = 5$. The energies are averaged over 100 independent node realizations. We can observe from Fig. 2 that there is a wide range of values of $a_{\rm th}$ such that the energy loss is practically independent of $a_{\rm th}$. This shows that our proposed procedure is

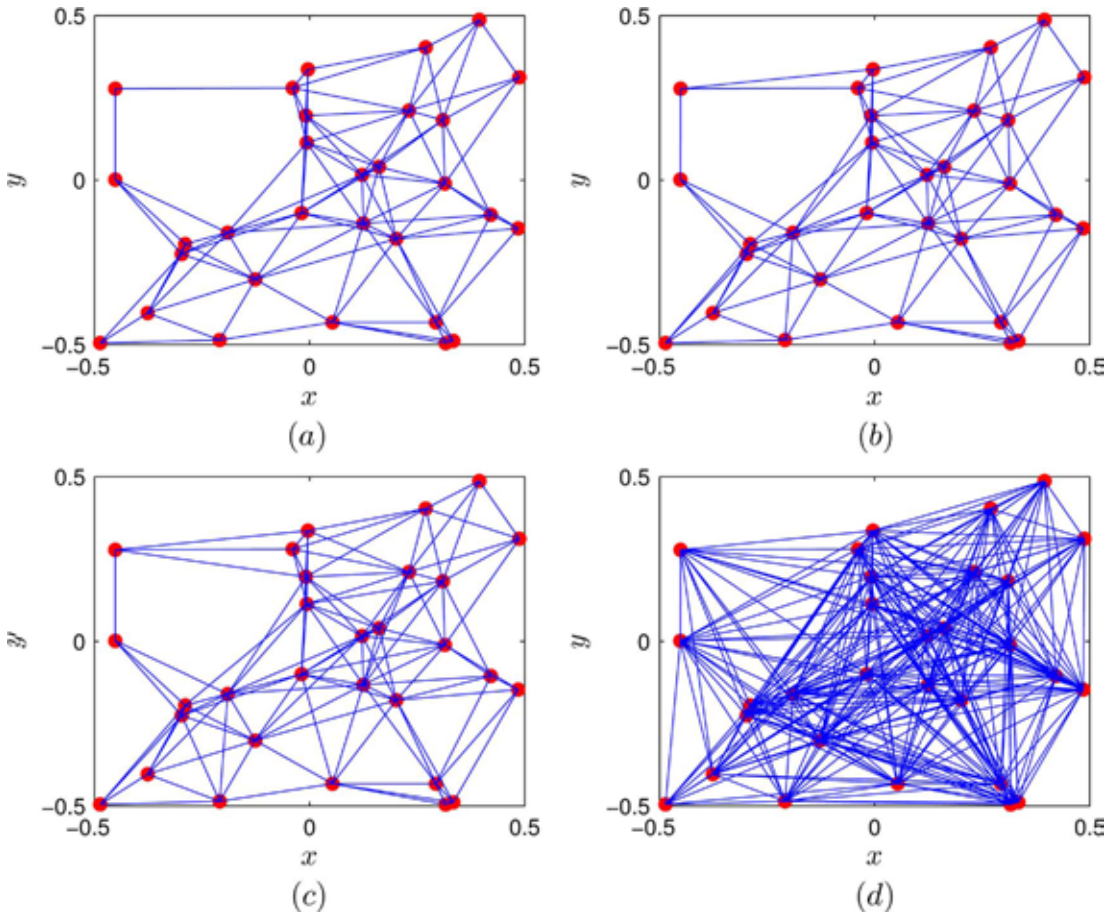

Fig. 3. Optimal topologies, for different threshold values and fixed $\eta(\eta = 6)$: (a) $a_{\rm th} = 0.09$. (b) $a_{\rm th} = 0.05$. (c) $a_{\rm th} = 5 \times 10^{-4}$. (d) $a_{\rm th} = 10^{-7}$.

rather robust with respect to the choice of $a_{\rm th}$. It is now interesting to check the effect of the threshold $a_{\rm th}$ on the topology, considering larger scale networks (in this case, we can only run our algorithm, as the exhaustive search is not feasible). As an example, in Fig. 3 we show the topologies obtained by solving problem $[\mathbf{P.3}]$, for a network composed of $n = 30$ nodes, using different values of $a_{\rm th}$, for $\eta = 6$. Comparing the four cases reported in Fig. 3, we notice that, only for very low values of the threshold [i.e., case (d)], we appreciate a sensitive change of topology, whereas for a large range of values of $a_{\rm th}$, the final topology is practically the same. This means that the proposed algorithm, in spite of the relaxation step and the subsequent quantization, yields rather stable solutions.

The previous results pertain to a specific realization of the node locations. To provide results of more general validity, in Fig. 4, we report the average value of: a) fraction of active links $\frac{\sum_{i=1}^{n}|\mathcal{N}_i|}{n(n-1)}$; b) $\lambda_2(\boldsymbol{L})$; and c) the average energy $\mathcal{E}_r$, as a function of the threshold $a_{\rm th}$. The averages are carried out over 100 independent realizations of the nodes location. From Fig. 4, we observe that there is an interval of values of $a_{\rm th}$ (roughly, between 10^{-5} to 10^{-2}) for which we obtain a strong reduction in the fraction of active links, with respect to the situation where there is no threshold, still achieving nearly the same performance, in terms of algebraic connectivity and energy consumption. This is indeed an important result, as it shows that the relaxed algorithm is weakly sensitive to the choice of $a_{\rm th}$.

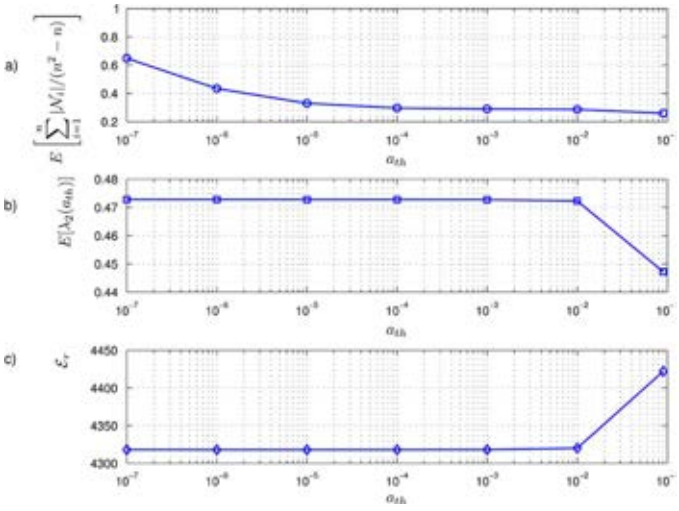

Fig. 4. Average value of (a) fraction of active links. (b) $\lambda_2(\boldsymbol{L})$. (c) energy versus threshold value for $\eta = 6$.

Example 3: Impact of Propagation Parameters on Final Topology: It is also interesting to look at the change in topology as a function of the radio communication model. To this end, in Fig. 5, we plot the optimal topologies achieved for the same node locations as in Fig. 3, but pertaining to different path loss exponents η, for a given threshold. Interestingly, we

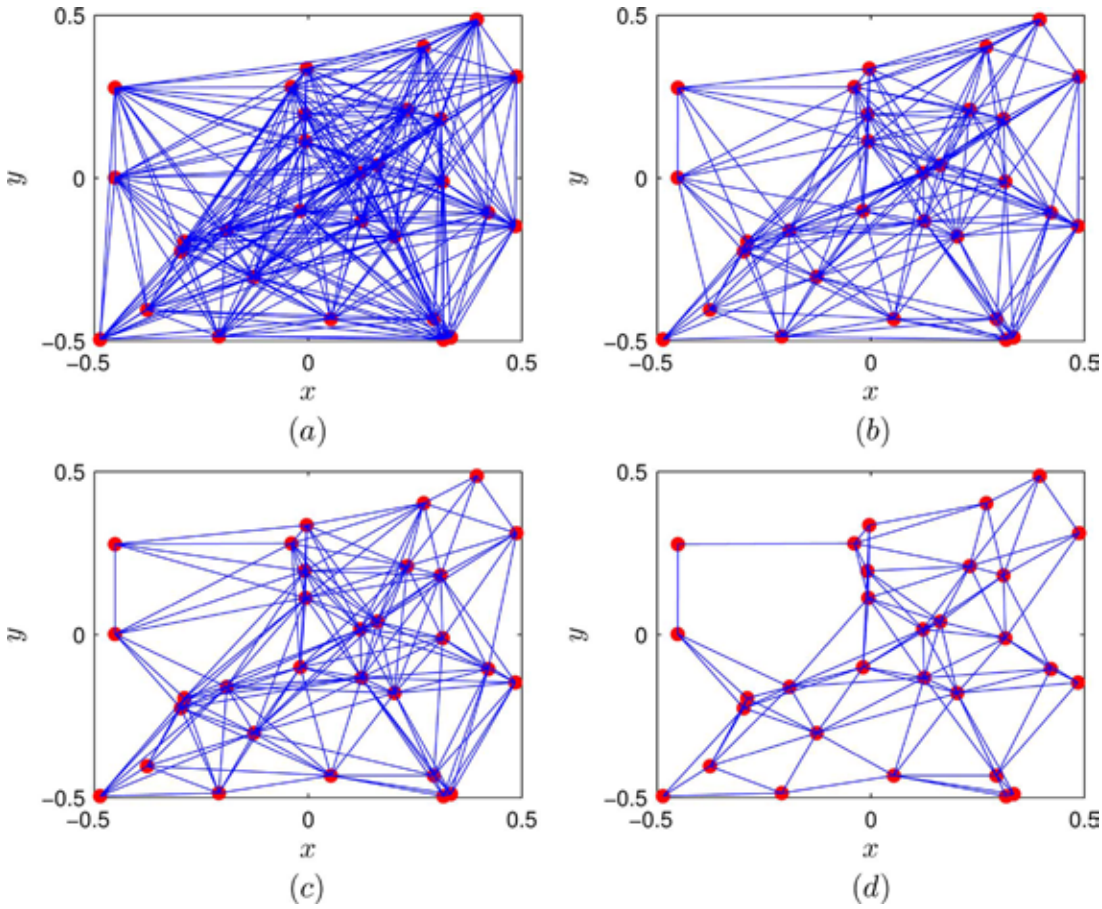

Fig. 5. Optimal topologies, for different path loss exponents η and fixed threshold ($a_{\rm th} = 0.09$): (a) $\eta = 3$. (b) $\eta = 4$. (c) $\eta = 5$. (d) $\eta = 6$.

notice, as expected, that, as the propagation loss increases (i.e., η increases), the network tends to become more and more sparse.

The results shown in Fig. 5 refer to a single realization of the node positions. To draw conclusions of more general validity, we averaged over 100 statistically independent realizations of the node locations, for networks of 30 nodes. In Fig. 6, we report the same performance metrics as in Fig. 4, but now as a function of the path loss exponent η, setting $a_{\rm th} = 0.09$. From Fig. 6, we observe that when the attenuation is high (i.e., η is large), reducing the number of links (making the topology sparser) is more important than reducing convergence time. Conversely, when the attenuation is low (i.e., η is small), increasing network connectivity is more important than reducing power consumption. This behavior sounds reasonable and in agreement with intuition.

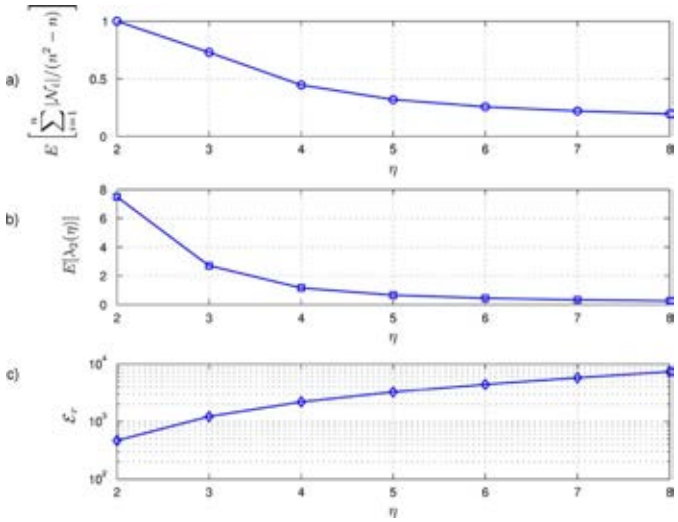

Fig. 6. Average value of (a) fraction of active links. (b) $\lambda_2(\boldsymbol{L})$. (c) energy versus path loss η for $a_{\rm th} = 0.09$.

V. Optimal Topology for Random Geometric Networks

In this section, we remove the assumption that the node locations are known a priori and model the network as a random geometric graph (RGG). In such a case, the graph connectivity properties and the convergence time can only be established in a probabilistic sense, asymptotically, as the number of nodes tends to infinity. We refer to [22] for the first basic result about the convergence of consensus algorithms over random graphs and to [23] for a more recent generalization of the convergence conditions under stochastic disturbances. As shown in [22], the rate of convergence of consensus algorithms in the random graph case is dictated by the expected value $E[e^{-2T_s\lambda_2(\boldsymbol{L})}]$. In the following, exploiting the concentration properties of the eigenvalues of RGGs [24], [25], we will show how to relate the convergence time to the expected value of $\lambda_2(\boldsymbol{L})$. This link

will be fundamental to derive the optimal coverage radius, and then transmit power, that minimize the energy consumption necessary to achieve consensus over RGGs. To achieve this goal, it is fundamental to recall and extend some results about the spectrum of RGGs.

A. RGGs

A random graph is obtained by distributing n points randomly over the d-dimensional space $\mathbb{R}^d$ and connecting the nodes according to a given rule. Let $V_n = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ be a set of d-dimensional vectors $\boldsymbol{x}_i$, belonging to a compact set $\Omega = [-1/2, 1/2]^d$ of $\mathbb{R}^d$, whose entries denote the coordinates of the nodes in $\mathbb{R}^d$, and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^d$. The graph topology is captured by the adjacency matrix $\boldsymbol{A}$ which, in this case, is a random matrix. An important class of random matrices, encompassing the adjacency matrix of our problem, is the so-called Euclidean Random Matrix (ERM) class, introduced in [26]. Given a set of n points located at positions $\boldsymbol{x}_i, i = 1, \ldots, n$, an $n \times n$ adjacency matrix $\boldsymbol{A}$ is an ERM if its generic (i, j) entry depends only on the difference $\boldsymbol{x}_i - \boldsymbol{x}_j$, i.e., $a_{ij} = F(\boldsymbol{x}_i - \boldsymbol{x}_j)$, where F is a measurable mapping from $\mathbb{R}^d$ to $\mathbb{R}$. An important subclass of ERM is given by the adjacency matrices of the so-called RGGs. In such a case, the entries a_{ij} of the adjacency matrix are either zero or one depending only on the distance between nodes i and j, i.e.

$$a_{ij} = \begin{cases} 1 & \text{if } \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leq r \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

where r is the coverage radius. This is a particular case of an ERM, corresponding to having

$$F(\boldsymbol{x}_i - \boldsymbol{x}_j) = \begin{cases} 1 & \text{if } \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \leq r \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

We will use the symbol $G(n, r)$ to indicate an RGG composed of n points, with coverage radius r.

The RGG model is the most appropriate to capture the topology of a wireless network, as it basically states that there is a link between two nodes only if they are within the coverage radius of each other[6].

Next, we recall some of the most important properties of RGG's, in terms of connectivity and spectrum, as they are relevant to our optimization problem.

1) Connectivity: Some interesting results on the asymptotic connectivity of random geometric graphs have been derived in a seminal work by Gupta and Kumar [27] who proved that, given a set of n points uniformly distributed within a unit square (i.e., $d = 2$), the graph is connected almost surely if the coverage radius behaves as

$$r_0(n) = \sqrt{\frac{\log(n) + c_n}{\pi n}}$$

with $c_n \to \infty$, as $n \to \infty$. Conversely, if $c_n \to -\infty$, the graph is disconnected almost surely. This means that the expression $\sqrt{\frac{\log(n)}{\pi n}}$ represents a threshold distance. In the following, we will often use the shorthand notation $r^+(n)$ to indicate the law $\sqrt{\frac{\log(n)+c_n}{\pi n}}$, with $c_n \to \infty$, as $n \to \infty$. Hence, a coverage radius $r_0(n)$ behaving as $r^+(n)$ represents a law that ensures connectivity with high probability, as $n \to \infty$. We will use the notation $r_0(n) \sim r^+(n)$ to indicate such a behavior.

In [4], [28], it has been shown that the degree of a RGG $G(n, r)$ of points uniformly distributed over a two-dimensional unit torus[7] is equal to

$$d(n) = \pi r^2 n \tag{22}$$

with high probability, i.e., with probability $1 - 1/n^2$, if the radius behaves as $r_0(n) \sim r^+(n)$. This implies that an RGG tends to behave, asymptotically, as a regular graph, if the coverage radius is chosen so as to guarantee connectivity with high probability.

We are primarily interested in the second eigenvalue of the Laplacian, $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{D}$ is the degree matrix and $\boldsymbol{A}$ is the adjacency matrix [see (1)]. From (22), $\boldsymbol{D} = \pi r^2 n \boldsymbol{I}$, so that we only need to investigate the second *largest* eigenvalue of $\boldsymbol{A}$. Hence, in the ensuing section, we study the spectrum of $\boldsymbol{A}$.

2) Spectrum of a Random Geometric Graph: In [24], [25], it is shown that the eigenvalues of the adjacency matrix, or of the transition probability matrix [8], tend to be concentrated, as the number of nodes tend to infinity. In particular, in [24] it is shown that the eigenvalues of the normalized adjacency matrix $\boldsymbol{A}_n = \boldsymbol{A}/n$ of an RGG $G(n, r)$, composed of points uniformly distributed over a unit bidimensional torus, tend to the Fourier series coefficients of the function F defined in (21)

$$\hat{F}(\boldsymbol{z}) = \int_{\Omega_r} \exp\left(-2\pi j \boldsymbol{z}^T \boldsymbol{x}\right) d\boldsymbol{x} \tag{23}$$

almost surely, for all $\boldsymbol{z} = [z_1, z_2] \in \mathbb{Z}^2$, where $\Omega_r = \{\boldsymbol{x} = [x_1, x_2]^T \in \mathbb{R}^2 : \|\boldsymbol{x}\| \leq r\}$. Using polar coordinates, i.e., $x_1 = \rho \sin\theta$ and $x_2 = \rho\cos\theta$, with $0 \leq \rho \leq r$ and $0 \leq \theta \leq 2\pi$, we obtain

$$\hat{F}(\boldsymbol{z}) = \int_0^r \int_0^{2\pi} \exp\left(-2\pi j \rho (z_1 \sin\theta + z_2 \cos\theta)\right) \rho d\rho d\theta.$$

This integral can be computed in closed form. Setting $z_1 = A\sin\phi$ and $z_2 = A\cos\phi$, we have

$$\hat{F}(A, \phi) = \int_0^r \int_{-\phi}^{2\pi - \phi} \exp\left(-2\pi j \rho A \cos(\xi)\right) \rho d\rho d\xi$$

with $\xi = \theta - \phi$. Furthermore, using the integral expression for the Bessel function of the first kind of order k, $J_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(jx\sin(\xi) - jk\xi\right) d\xi$, we get

$$\hat{F}(A, \phi) = \hat{F}(A) = 2\pi \int_0^r J_0(2\pi\rho A) \rho d\rho.$$

Finally, using the identity $\int_0^u v J_0(v) dv = u J_1(u)$, we can make explicit the dependence of $\hat{F}(A)$ on the index pair $[z_1, z_2]$

$$\hat{F}(z_1, z_2) = \frac{r}{\sqrt{z_1^2 + z_2^2}} J_1\left(2\pi r \sqrt{z_1^2 + z_2^2}\right). \tag{24}$$

[6] In practice, a wireless channel is also affected by fading, multipath, and shadowing. Hence, the presence of a link between two nodes depends on two sources of randomness: distance and fading. In this work, we concentrate on the single source of randomness, given by the distance between the nodes, but considering both sources of randomness will be an interesting extension of this paper.

[7] A torus geometry is typically used to get rid of border effects.

[8] The transition probability matrix is the adjacency matrix, normalized with respect to the node degree, so that the ith row of the adjacency matrix is divided by the degree of node i.

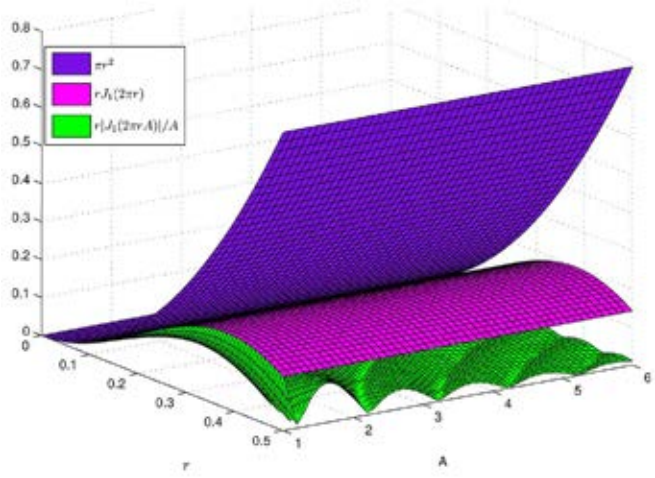

Fig. 7. Verification of the inequalities in (26).

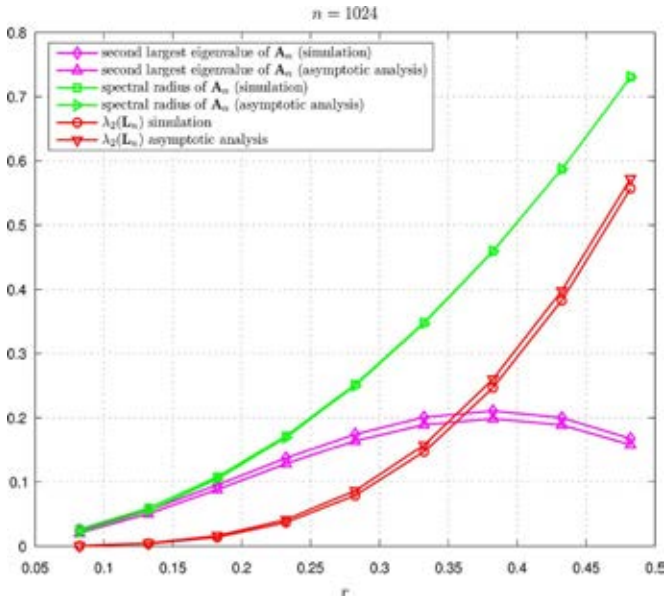

Fig. 8. Asymptotic analysis and simulation results for some eigenvalues of $\boldsymbol{A}_n$ and for $\lambda_2(\boldsymbol{L}_n)$ versus the transmission radius.

This formula allows us to rank the eigenvalues of $\boldsymbol{A}_n = \boldsymbol{A}/n$. In particular, we are interested in the second largest eigenvalue of $\boldsymbol{A}_n$. Considering that the minimum coverage radius ensuring connectivity behaves as $r(n) \sim \sqrt{\frac{\log(n)}{n}}$, i.e., it is a vanishing function of n, we can use the Taylor series expansion of $\hat{F}(z_1, z_2)$, for small r. Recalling that, for small x, $J_1(x) = x/2 - x^3/16 + o(x^5)$, we can approximate the eigenvalues as

$$\hat{F}(z_1, z_2) = \pi r^2 - \frac{\pi^3 (z_1^2 + z_2^2)\, r^4}{2} + o(r^6). \qquad (25)$$

This expansion shows that, at least for small r, the largest eigenvalue equals πr^2 and occurs at $z_1 = z_2 = 0$, whereas the second largest eigenvalue corresponds to the cases $(z_1 = 1, z_2 = 0)$ and $(z_1 = 0, z_2 = 1)$. More generally, we can check numerically that, for $r \leq 1/2$ and $A \geq 1$, the following inequalities hold true:

$$\pi r^2 \geq r J_1(2\pi r) \geq \frac{r}{A} |J_1(2\pi r A)|. \qquad (26)$$

The validity of these inequalities can be verified from Fig. 7, which shows the three terms in (26) as a function of r and A.

In summary, denoting the spectral radius of $\boldsymbol{A}_n$ as $\zeta_1(\boldsymbol{A}_n) = \max_{1 \leq i \leq n} \frac{|\lambda_i(n)|}{n}$, where $\{\lambda_i(n)\}_{i=1}^n$ is the set of eigenvalues of $\boldsymbol{A}$, it follows that

$$\lim_{n \to \infty} \zeta_1(\boldsymbol{A}_n) = \max_{\boldsymbol{z} \in \mathbb{Z}^2} \mid \hat{F}(\boldsymbol{z}) \mid = \hat{F}(0,0) = \pi r^2 \qquad (27)$$

while the second largest eigenvalue of $\boldsymbol{A}_n$, $\zeta_2(\boldsymbol{A}_n)$, converges to

$$\lim_{n \to \infty} \zeta_2(\boldsymbol{A}_n) = \hat{F}(1,0) = \hat{F}(0,1) = r J_1(2\pi r). \qquad (28)$$

We are now able to derive the asymptotic expression for the second largest eigenvalue of the normalized Laplacian $\boldsymbol{L}_n = \boldsymbol{D}_n - \boldsymbol{A}_n$, where $\boldsymbol{D}_n := \boldsymbol{D}/n$ is the normalized degree matrix. Because of the asymptotic property of the degree of an RGG, shown in (22), the second largest eigenvalue of $\boldsymbol{L}_n$ tends asymptotically to

$$\lambda_2(\boldsymbol{L}_n) = \pi r^2 - \zeta_2(\boldsymbol{A}_n). \qquad (29)$$

Thus, the algebraic connectivity of the graph can be approximated, asymptotically, as

$$\lambda_2(\boldsymbol{L}) = \pi n r^2 - n r J_1(2\pi r). \qquad (30)$$

Since the previous expressions have been derived in the asymptotic regime, in Fig. 8 we compare the analytic formulas for the first and the second largest eigenvalues of $\boldsymbol{A}_n$, as given in (27) and (28), with the numerical results obtained by averaging over 100 independent realizations of RGG's composed of $n = 1024$ nodes. We can notice the good agreement in Fig. 8 between the theoretical expression for the algebraic connectivity $\lambda_2(\boldsymbol{L}_n)$, given in (29), and the simulation results.

B. Minimization of the Energy Consumption: An Analytic Approach

We can now exploit the previous analytic expressions to study the energy minimization problem for RGG's. In the random topology case, since the distances are unknown, we cannot optimize the power associated with each link. However, we can seek the common transmit power that minimizes energy consumption. Thus, in the random setting we assume a broadcast communication model, where each node broadcasts the value to be shared with its neighbors. In the lack of any information about distances among the nodes, we assume that each node uses the same transmit power. In this case, the network topology can be modeled as a random graph model. It can be shown [22], [23] that the system in (3) converges to consensus almost surely, i.e., $\Pr\{\lim_{t \to \infty} \boldsymbol{x}(t) = x^* \boldsymbol{1}\} = 1$ assuming that each node has a coverage radius so that the network is asymptotically connected

with probability one [14]. Then the rate of convergence of the dynamical system in (3) is given [22], [23] by $E[e^{-2T_s\lambda_2(\boldsymbol{L})}]$. Note that defining $f(\lambda_2) = e^{-2T_s\lambda_2(\boldsymbol{L})}$ and using the Taylor series expansion of $f(\lambda_2)$[9] at the point $\lambda_2^0 = E[\lambda_2] = m_{\lambda_2}$, we can write

$$f(\lambda_2) = f(m_{\lambda_2}) + f^{(1)}(m_{\lambda_2})(\lambda_2 - m_{\lambda_2}) + \frac{f^{(2)}(m_{\lambda_2})}{2}(\lambda_2 - m_{\lambda_2})^2 + \sum_{k=3}^{\infty} \frac{f^{(k)}(m_{\lambda_2})}{k!}(\lambda_2 - m_{\lambda_2})^k. \quad (31)$$

Consequently, taking the expected value, we get

$$E[f(\lambda_2)] = f(m_{\lambda_2}) + \frac{f^{(2)}(m_{\lambda_2})}{2}\sigma_{\lambda_2}^2 + \dots \quad (32)$$

denoting with $\sigma_{\lambda_2}^2 = E[(\lambda_2 - m_{\lambda_2})^2]$ the variance of λ_2. But since all central moments of order greater than one of the eigenvalues tend to zero, because of the concentration property, we can use the approximation

$$E[e^{-2T_s\lambda_2}] \approx e^{-2T_s E[\lambda_2]}. \quad (33)$$

As a consequence, the energy spent to achieve consensus can now be approximated as

$$\mathcal{E} = K\frac{np}{2E[\lambda_2(\boldsymbol{L}(p))]}. \quad (34)$$

This is the performance metric we wish to minimize in the random scenario, with respect to the single unknown p.

In particular, using the asymptotic expression (30) for the algebraic connectivity, we can introduce the following metric

$$\mathcal{E}(r) = \frac{np_{\min}[1 + (r/r_0)^\eta]}{n\pi r^2 - rnJ_1(2\pi r)}. \quad (35)$$

We now check, numerically, that the function $\mathcal{E}(r)$ given in (35) is a convex function of r, for $r_0(n) \le r \le 0.5$, where $r_0(n) \sim r^+(n)$, to ensure connectivity.

Let us rewrite (35) as

$$\mathcal{E}(r) = \frac{g(r)}{\lambda_2(r)}$$

with $g(r) = np_{\min}[1 + (r/r_0)^\eta]$ and $\lambda_2(r)$ as in (30). The first– and second–order derivatives of $g(r)$ are, respectively

$$g^{'}(r) = \frac{dg(r)}{dr} = np_{\min}\eta\frac{r^{\eta-1}}{{r_0}^\eta} > 0$$
$$g^{''}(r) = \frac{d^2g(r)}{dr^2} = np_{\min}\eta(\eta-1)\frac{r^{\eta-2}}{{r_0}^\eta} > 0$$

[9]For simplicity we drop in λ_2 the dependence on $\boldsymbol{L}$.

so that $g(r)$ is a convex increasing positive function of r. Let us now study the behavior of $\lambda_2(r)$. Using

$$\frac{d[r^m J_m(r)]}{dr} = r^m J_{m-1}(r) \quad \text{for} \quad m = 0, 1, 2, \dots$$

we obtain

$$\lambda_2^{'}(r) = \frac{d\lambda_2(r)}{dr} = 2\pi rn[1 - J_0(2\pi r)]$$

with $\lambda_2^{'}(r) > 0$ for $r_0(n) \le r \le 0.5$. Moreover, since

$$\frac{dJ_m(r)}{dr} = \frac{m}{r}J_m(r) - J_{m+1}(r) \quad \text{for} \quad m = 0, 1, 2, \dots$$

we have

$$\lambda_2^{''}(r) = \frac{d^2\lambda_2(r)}{dr^2} = 2\pi n[1 - J_0(2\pi r)] + 4\pi^2 rnJ_1(2\pi r). \quad (36)$$

Observe that the first term on the right-hand side (RHS) of (36) is always positive. Furthermore, if $r \le 1/2$, the second term is also positive, since $J_1(2\pi r) > 0$ for $r \le 1/2$. Hence, $\lambda_2^{''}(r) > 0$ and we can conclude that the algebraic connectivity is an increasing and convex function of r for $r_0(n) \le r \le 0.5$, where $r_0(n) \sim r^+(n)$.

We can now compute the first- and second-order derivatives of the energy function with respect to r. We get

$$\mathcal{E}^{'}(r) = \frac{g^{'}(r)}{\lambda_2(r)} - \frac{\lambda_2^{'}(r)g(r)}{\lambda_2^2(r)}$$

and, substituting the corresponding expressions, the extremal points can be obtained by solving the following nonlinear equation:

$$\mathcal{E}^{'}(r) = a_1(r) + a_2(r)J_0(2\pi r) - a_3(r)J_1(2\pi r) = 0$$

with $a_1(r) = \pi[(\eta-2)r^\eta - 2r_0^\eta]$, $a_2(r) = 2\pi(r^\eta + {r_0}^\eta)$ and $a_3(r) = \eta r^{\eta-1}$. Furthermore, the second derivative of $\mathcal{E}(r)$ is given by the equation shown at the bottom of the page. In Fig. 9, we report $\mathcal{E}^{''}(r)$ as a function of r, for different values of η. From Fig. 9, we can check that the second derivative is always positive in the range of interest. This verifies that $\mathcal{E}(r)$ is indeed a convex function of r defined on the compact convex set $r_0(n) \le r \le 0.5$. As a consequence, we can state that there is always at least a radius r that globally minimizes the energy consumption in a RGG.

Numerical examples. In Fig. 10, we compare the value of $\mathcal{E}(r)$ obtained by our theoretical approach and by simulation, for various values of the path loss exponent, $2 \le \eta \le 6$. The results are averaged over 100 independent realizations of random geometric graphs composed of $n = 1000$ nodes. For each η, the pair of radius and energy providing minimum energy consumption is indicated by a circle (simulation) or a star (theory). We observe

$$\mathcal{E}^{''}(r) = \frac{(g^{''}(r)\lambda_2(r) - 2\lambda_2^{'}(r)g^{'}(r))\lambda_2(r) + g(r)(2{\lambda_2^{'}}^2(r) - \lambda_2^{''}(r)\lambda_2(r))}{\lambda_2^3(r)}.$$

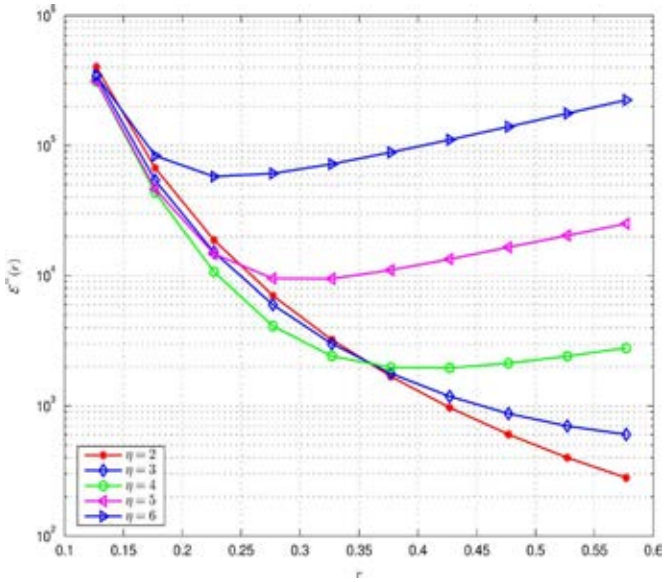

Fig. 9. Second derivative of the energy versus the transmission radius for several values of η.

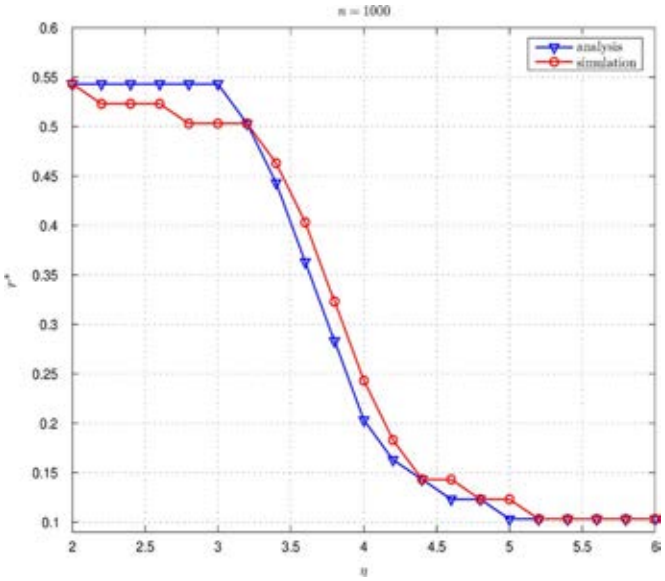

Fig. 11. Optimal transmission radius versus η.

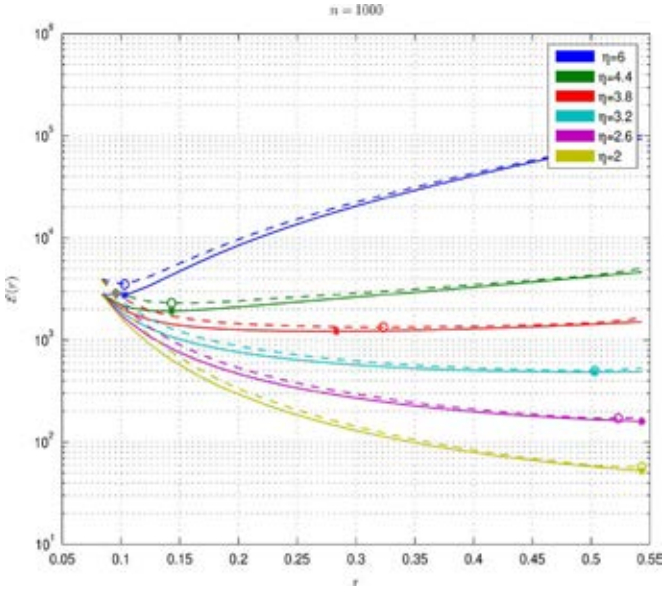

Fig. 10. Global energy consumption versus transmission radius for an RGG; theoretical values (solid) and simulation (dashed).

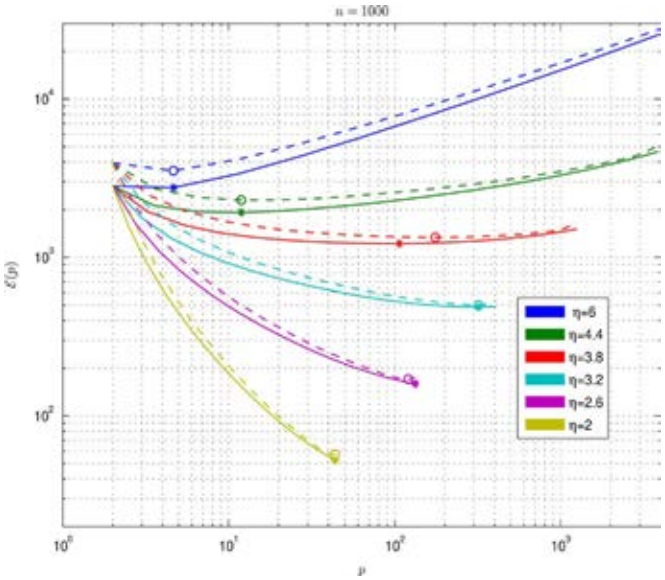

Fig. 12. Global energy consumption versus per node transmit power for an RGG; theoretical values (solid) and simulation (dashed).

that the theoretical derivations provide a very good prediction of the performance achieved by simulation. Furthermore, for each η, there is a coverage radius value that minimizes energy consumption.

The optimal values of the coverage radius as a function of η, as predicted by our theoretical derivations or by simulation, are reported in Fig. 11. From this figure, we observe that there is a clear transition from the low power attenuation regime (i.e., $\eta < 3.5$), where the optimal radius tends to make the network fully connected, as opposed to the strong attenuation situation ($\eta > 4.5$), where the network is minimally connected. This means that in the low attenuation case, minimizing convergence time is more important than minimizing power consumption. Conversely, in the large attenuation case, it is more important to minimize power consumption, by limiting the number of links as much as possible, compatibly with the constraint of ensuring connectivity.

Fig. 12 shows the average energy consumption versus the per node transmit power p, under the same settings of Fig. 10. The circles (simulation) and the stars (theory) represent, again, the values of p that minimize energy consumption, for each η. It can verified the existence of an optimal transmit power value minimizing the energy consumption.

Finally, Fig. 13 shows the optimal per node power versus η. For low values of η, the transmit power increases with η because it must cope with higher attenuations to guarantee connectivity. Conversely, for large values of η, the optimal power decreases with η because, in such a case, it is more beneficial to limit

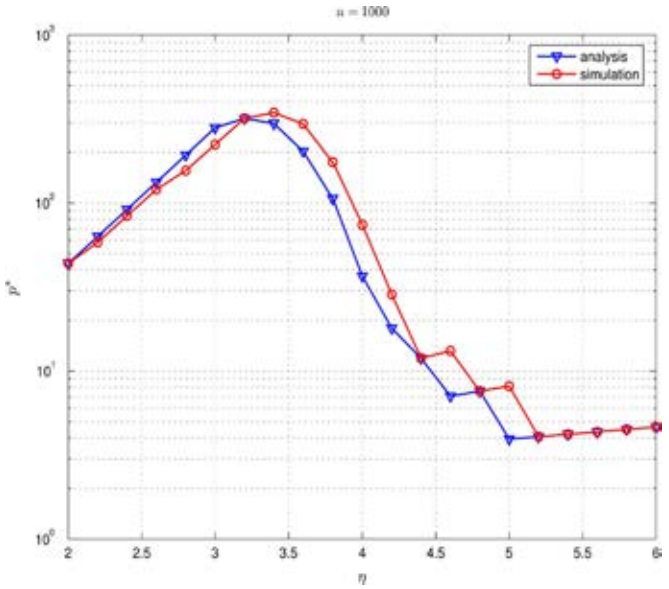

Fig. 13. Optimal per node transmit power versus η.

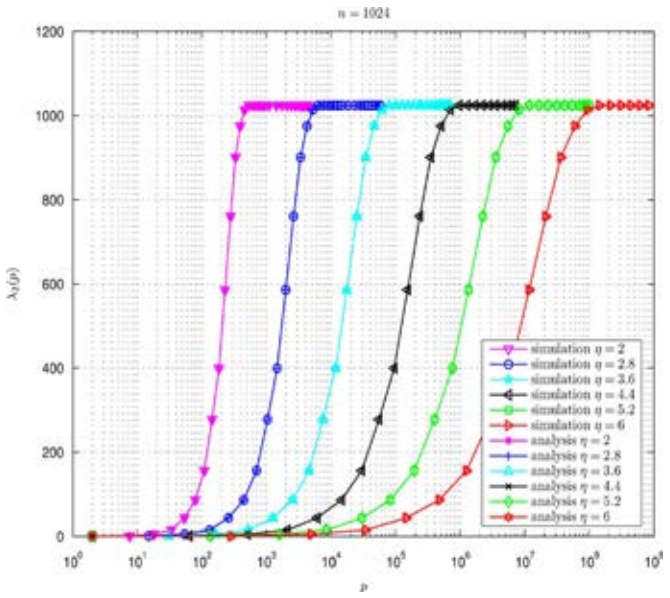

Fig. 14. Network algebraic connectivity versus per node transmit power for several values of the path loss coefficient.

the number of links, until the minimum power guaranteeing the connectivity constraint is reached.

C. Random Topologies versus Uniformly Spaced Grids

Finally, it is interesting to compare the energy consumption achievable with a random geometric graph and a deterministic grid. In the deterministic case, we may also distinguish between a graph with nodes scattered arbitrarily in a given area, but with positions known, and a regular uniform grid, where the points are located over a rectangular grid.

1) Eigenvalues of a Planar Uniform Grid: We start by deriving the algebraic connectivity of a square grid whose n nodes are uniformly spaced within a unit square, at a distance $\delta = 1/n_1 = 1/\sqrt{n}$. To avoid undesired border effects, we consider the wrapping of the unit square in order to form a toroidal surface. Each node is assumed to have a link with the neighboring nodes only if they are at a distance less than a coverage radius $r(p)$, that depends on the transmit power p, as with geometric graphs. In Appendix A, we derive a closed form expression for the network degree and for the algebraic connectivity [see, e.g., (45)–(48)].

A numerical check of our derivations is reported in Fig. 14, where we show the numerical value of $\lambda_2(\boldsymbol{L})$, obtained through the eigendecomposition of $\boldsymbol{L}$, and the value given in (48), for different values of η. From Fig. 14, we can see a perfect agreement between our closed form expression and the numerical results. Notice, in particular, the sharp transition behavior: the eigenvalues tend to n or 0 depending on whether the transmit power is above or below a threshold; further, the threshold increases with η.

In Fig. 15, we compare the algebraic connectivity of the rectangular grid, given by (48), with the theoretical value obtained for the random geometric graph, given in (30), assuming the same node density, over the same toroidal surface. As shown in previous works, see, e.g., [4], RGGs tend to behave asymptotically as a regular graph. The result shown in Fig. 15 is a further

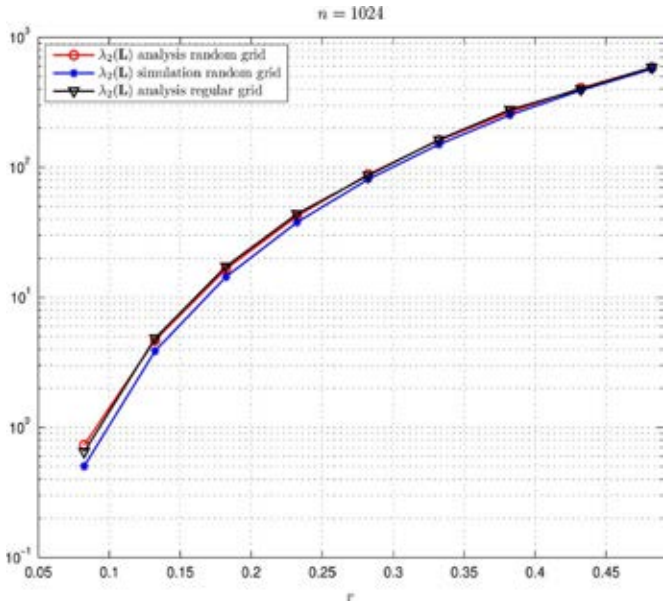

Fig. 15. Algebraic connectivity of random geometric graph and rectangular grid versus transmission radius.

confirmation of this property. In the same figure, we also report simulation results for the algebraic connectivity, obtained by averaging over 100 independent realizations of a RGG. We can check, once again, the good agreement between theory and simulation.

Finally, in Fig. 16 we compare the energy consumption obtained assuming full *a priori* knowledge of the nodes' locations or no knowledge at all. In the first case, the topology and the power allocation over each link are optimized according to the method illustrated in Section IV. The optimal values, for each η, are indicated by colored dots. In the second case, we report (solid line) the energy consumption versus the average power, assuming that all nodes transmit with the same power (since they

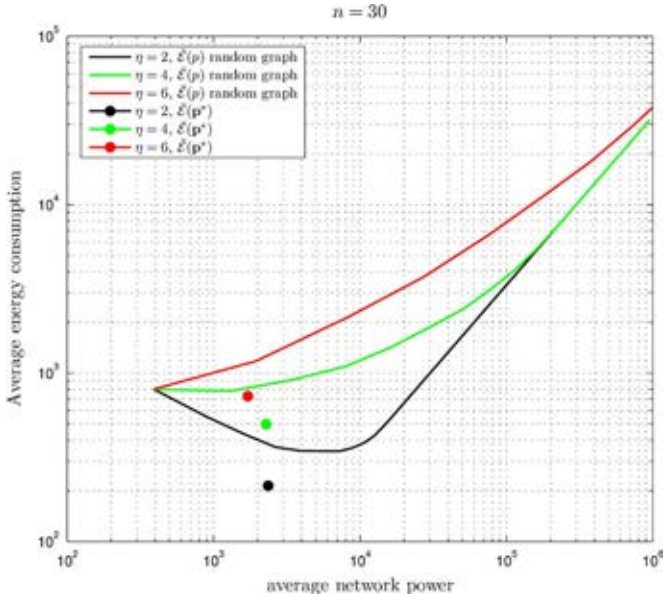

Fig. 16. Average network energy consumption versus average network power.

do not have any prior information about the other nodes positions). To make a fair comparison between the two cases, we consider in both cases the sum of the powers associated with each active link. This means that, in the second case, the energy consumption is measured, for every common transmit power p, as $\mathcal{E}(p) = \frac{p\sum_{i=1}^{n} n_i(p)}{\lambda_2(p)}$, where $n_i(p)$ denotes the number of neighbors of node i. The results shown in Fig. 16 have been obtained by averaging the results obtained in the two settings over the same set of 100 independent random deployments of $n = 30$ nodes, uniformly distributed over the unit square.

From Fig. 16 we can see that, as expected, the method assuming perfect *a priori* knowledge of the node positions (in the figure denoted as $\bar{\mathcal{E}}(\boldsymbol{p}^*)$) is indeed able to achieve better performance than the other method ($\bar{\mathcal{E}}(p)$), as it is able to take advantage of the *a priori* knowledge. The optimal average power is the result of a tradeoff between transmit power and convergence time, and always depends on the path loss exponent.

VI. Conclusion

In this paper we have addressed the problem of finding the network topology that minimizes the energy consumption necessary to achieve consensus in a WSN. Assuming a simple flat fading propagation model, we have studied two main network models: 1) arbitrary networks, where the nodes are arbitrarily located, but with known positions, and 2) a random model, where the positions are not known a priori, but are modeled as random variables. In the first case, we have shown how to optimize the network topology and the power allocated to each active link in order to minimize the total energy necessary to achieve consensus within a prescribed accuracy. Topology optimization is, in general, a combinatorial problem and hence computationally demanding. To simplify the solution, we introduced a relaxation step that enabled us to reformulate the energy minimization problem as a convex-concave fractional program. This alternative formulation can be cast as an equivalent parametric convex problem, which enables efficient solutions. The link weights resulting from the solution of the convex problem must then be thresholded to find out the network topology. We have shown through numerical results that the thresholding operation may yield a considerable reduction of the number of active links, yet with very limited effect on performance.

In our deterministic setting, we assumed point-to-point links and we optimized the power over each link. In practice, a consensus algorithm running over a wireless network could benefit from the broadcast channel. Deciding between broadcast or one-to-one links entails a proper choice of the medium access strategy, to establish in which time slot each node has to listen to which broadcaster.

Conversely, in the random network case, lacking any information about the internode distances, we assumed a broadcast communication strategy. In such a case, the network topology is modeled as a random geometric graph. We have derived closed form expressions, albeit valid only asymptotically, for the algebraic connectivity, assuming a common transmit power. Then, building on these expressions, we have shown that the energy consumption is a convex function of the coverage radius. We have also shown that a random geometric graph performs, asymptotically, as a regular graph built over a rectangular grid. This confirms previous results, although now in the context of energy minimization over consensus networks.

Finally, we have compared the performance achievable with arbitrary and random graphs. Clearly, the knowledge of the node locations allows better power allocation, that translates into lower energy consumption to achieve consensus. However, in practice there is a price associated with the knowledge of node location. This knowledge requires the acquisition of the node positions first and then a centralized optimization. Conversely, the random approach can be followed also in a decentralized fashion, with only minimal information about some global parameters like number of nodes and area covered by the network, and it does not need any extra hardware or computation to acquire the nodes' locations.

In this paper, we have assumed a simple flat fading channel model, whose effect is only to introduce attenuation and superimpose noise. The simple model captures the essence of the problem and keeps the overall problem complexity under control. However, looking at potential applications, it would be interesting to generalize the approach to the case where the channel model is more complicated. Furthermore, the network topology has been assumed to be static. However, a dynamic topology, possibly adapted to the consensus state, might provide better performance.

Appendix A

In this section, we derive an analytical expression for the eigenvalues of the adjacency matrix $\boldsymbol{A}$ of a rectangular grid over a unit torus, as a function of the coverage radius $r(p)$. We assume, for simplicity that the number of nodes n is a square number, i.e., $n = n_1^2$, with n_1 integer. We number the rows of

the grid, 1 to n_1, going from bottom to top. The $n \times n$ matrix $\boldsymbol{A}$ can be expressed in a cyclic block form, as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \dots & \boldsymbol{A}_{n_1-2} & \boldsymbol{A}_{n_1-1} \\ \boldsymbol{A}_{n_1-1} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \dots & \boldsymbol{A}_{n_1-3} & \boldsymbol{A}_{n_1-2} \\ \boldsymbol{A}_{n_1-2} & \boldsymbol{A}_{n_1-1} & \boldsymbol{A}_0 & \dots & \boldsymbol{A}_{n_1-4} & \boldsymbol{A}_{n_1-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \dots & \boldsymbol{A}_{n_1-1} & \boldsymbol{A}_0 \end{bmatrix}$$

where each block $\boldsymbol{A}_i$, for $i = 0, \dots, n_1 - 1$, is an $n_1 \times n_1$ circulant matrix. In particular, the block $\boldsymbol{A}_0$ is the adjacency matrix between the nodes lying on the same row of the grid, starting from the leftmost node and proceeding along the row. $\boldsymbol{A}_i$ represents the adjacency matrix between the nodes of row j and the nodes lying on the $(i+j)$th row, for $i = 1, \dots, n_1 - j$, and on the $(i + j - n_1)$th row for $i = n_1 - j + 1, \dots, n_1 - 1$. The toroidal structure, used to avoid border effects, reflects into the cyclic structure of $\boldsymbol{A}$, so that we have $\boldsymbol{A}_{n_1-i} = \boldsymbol{A}_i$ if

$$\begin{aligned} &i = (n_1 - 1)/2 + 1, \dots, n_1 - 1 \quad \text{and} \quad n_1 \text{ odd} \\ &i = n_1/2 + 1, \dots, n_1 - 1 \quad \text{and} \quad n_1 \text{ even}. \end{aligned}$$

We also note that $\boldsymbol{A}$ is a block circulant matrix with circulant blocks (BCCB). Since each block $\boldsymbol{A}_i$ for $i = 0, \dots, n_1 - 1$ is an $n_1 \times n_1$-dimensional circulant matrix, the eigenvalues of $\boldsymbol{A}_i$ admit the following decomposition

$$\boldsymbol{A}_i = \tilde{\boldsymbol{F}}_{n_1}^H \boldsymbol{\Lambda}_i \tilde{\boldsymbol{F}}_{n_1}$$

with

$$\tilde{\boldsymbol{F}}_{n_1} = \frac{1}{\sqrt{n_1}} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n_1-1} & \omega^{n_1-2} & \dots & \omega \end{bmatrix}$$

where $\omega = \exp(-j2\pi/n_1)$; $\boldsymbol{\Lambda}_i$ is a diagonal matrix whose entries are the eigenvalues of $\boldsymbol{A}_i$, i.e., the discrete Fourier transform (DFT) of the first row of $\boldsymbol{A}_i$. Hence, the matrix $\boldsymbol{A}$ admits the following diagonalization:

$$\boldsymbol{A} = (\tilde{\boldsymbol{F}}_{n_1} \otimes \tilde{\boldsymbol{F}}_{n_1})^H \boldsymbol{\Lambda} (\tilde{\boldsymbol{F}}_{n_1} \otimes \tilde{\boldsymbol{F}}_{n_1})$$

where $\otimes$ denotes the Kronecker product and

$$\boldsymbol{\Lambda} = \sum_{k=0}^{n_1-1} \boldsymbol{\Omega}_{n_1}^k \otimes \boldsymbol{\Lambda}_k$$

with $\boldsymbol{\Omega}_{n_1} = \mathrm{diag}[1, \omega, \dots, \omega^{n_1-1}]$. The entries $\lambda_{l,m}(\boldsymbol{A})$ of the matrix $\boldsymbol{\Lambda}$ can be also expressed, after a few algebraic manipulations, as

$$\lambda_{l,m}(\boldsymbol{A}) = \sum_{i=0}^{n_1-1} \sum_{k=0}^{n_1-1} a_k^{(i)}(p) \times \exp(-j2\pi kl/n_1) \exp(-j2\pi mi/n_1) \quad (37)$$

for $l, m = 0, \dots, n_1 - 1$ and the coefficient $a_k^{(i)}(p)$ is kth entry of the first row of $\boldsymbol{A}_i$ for $k, i = 0, \dots, n_1 - 1$. Equivalently, (37) can be seen as the two-dimensional DFT of a matrix $\tilde{\boldsymbol{A}}$ whose rows $\tilde{\boldsymbol{A}}(i,:)$ are given by $\tilde{\boldsymbol{A}}(i,:) = \boldsymbol{A}_i(1,:)$ for $i = 0, \dots, n_1 - 1$, applying the 1-DFT to the rows and then to the columns of $\tilde{\boldsymbol{A}}$.

In general, for a regular grid where each node has degree $d(p)$, the algebraic connectivity can be expressed as

$$\lambda_2(\boldsymbol{L}(p)) = d(r(p)) - \sum_{i=0}^{n_1-1} \sum_{k=0}^{n_1-1} a_k^{(i)}(p) \exp(-j2\pi k/n_1) \quad (38)$$

where the coefficients $a_k^{(i)}(p)$ can be expressed as follows. Let $u(x)$ denote the step function, i.e., $u(x) = 1$ for $x > 0$ and zero otherwise. For n_1 odd, we have

$$\begin{aligned} &a_0^{(0)}(p) = 0 \quad \text{and} \\ &a_0^{(i)}(p) = u(r(p) - i\delta) \quad i = 1, \dots, (n_1 - 1)/2, \quad (39) \\ &a_k^{(0)}(p) = u(r(p) - k\delta) \quad \text{and} \\ &a_k^{(i)}(p) = u(r(p) - \sqrt{i^2 + k^2}\delta) \quad i, k = 1, \dots, (n_1 - 1)/2 \quad (40) \end{aligned}$$

$$a_k^{(i)}(p) = a_{n_1-k}^{(i)}(p) \quad k = (n_1 - 1)/2 + 1, \dots, n_1 - 1 \quad \text{and} \quad i = 0, \dots, (n_1 - 1)/2. \quad (41)$$

Finally $\boldsymbol{A}_i = \boldsymbol{A}_{n_1-i}$ for $i = (n_1 - 1)/2 + 1, \dots, n_1 - 1$. Similarly, in the case of n_1 even, we have

$$\begin{aligned} &a_0^{(0)}(p) = 0 \quad \text{and} \\ &a_0^{(i)}(p) = u(r(p) - i\delta) \quad i = 1, \dots, n_1/2, \quad (42) \\ &a_k^{(0)}(p) = u(r(p) - k\delta) \quad \text{and} \\ &a_k^{(i)}(p) = u(r(p) - \sqrt{i^2 + k^2}\,\delta) \quad i, k = 1, \dots, n_1/2, \quad (43) \end{aligned}$$

$$a_k^{(i)}(p) = a_{n_1-k}^{(i)}(p) \quad k = n_1/2 + 1, \dots, n_1 - 1 \quad \text{and} \quad i = 0, \dots, n_1/2 \quad (44)$$

with $\boldsymbol{A}_i = \boldsymbol{A}_{n_1-i}$ if $i = n_1/2 + 1, \dots, n_1 - 1$.

Finally, we can derive analytical expressions of the network degree and of the algebraic connectivity for n_1 odd as

$$d(p) = \sum_{k=1}^{(n_1-1)/2} 2a_k^{(0)}(p) + 2 \sum_{i=1}^{(n_1-1)/2} \left(a_0^{(i)}(p) + 2 \sum_{k=1}^{(n_1-1)/2} a_k^{(i)}(p) \right) \quad (45)$$

and

$$\lambda_2(\boldsymbol{L}(p)) = \sum_{k=1}^{(n_1-1)/2} 2a_k^{(0)}(p)(1 - \cos(2\pi k/n_1)) + 4 \sum_{i=1}^{(n_1-1)/2} \sum_{k=1}^{(n_1-1)/2} a_k^{(i)}(p)(1 - \cos(2\pi k/n_1)) \quad (46)$$

and for n_1 even

$$\begin{aligned} d(p) = & \sum_{k=1}^{n_1/2-1} 2\left(a_k^{(0)}(p) + a_k^{(n_1/2)}(p)\right) \\ & + \sum_{i=1}^{n_1/2-1} 2\left(a_{n_1/2}^{(i)}(p) + a_0^{(i)}(p)\right) \\ & + a_{n_1/2}^{(0)}(p) + a_0^{(n_1/2)}(p) \\ & + a_{n_1/2}^{(n_1/2)}(p) + 4\sum_{i=1}^{n_1/2-1}\sum_{k=1}^{n_1/2-1} a_k^{(i)}(p) \end{aligned} \quad (47)$$

and

$$\begin{aligned} \lambda_2(\boldsymbol{L}(p)) = & 4\sum_{i=1}^{n_1/2-1}\sum_{k=1}^{n_1/2-1} a_k^{(i)}(p) \\ & + \sum_{k=1}^{n_1/2-1} 2\left(a_k^{(0)}(p) + a_k^{(n_1/2)}(p)\right) \\ & \times (1-\cos(2\pi k/n_1)) + 4\sum_{i=1}^{n_1/2-1} a_{n_1/2}^{(i)}(p) \\ & + 2a_{n_1/2}^{(0)}(p) + 2a_{n_1/2}^{(n_1/2)}(p) \\ & - 4\sum_{i=1}^{n_1/2-1}\sum_{k=1}^{n_1/2-1} a_k^{(i)}(p)\cos(2\pi k/n_1). \end{aligned} \quad (48)$$

For example, in the case where $r(p) < \sqrt{2}\delta$, i.e., assuming a regular grid of degree $d(p) = 4$, we obtain

$$\boldsymbol{A}_{n_1-1} = \boldsymbol{A}_1 = \boldsymbol{I} \quad \text{and}\, \boldsymbol{A}_i = \boldsymbol{O}\, \forall\, i = 2, \ldots, n_1 - 2 \quad (49)$$

while $\boldsymbol{A}_0$ is a circulant matrix whose first row is the n_1-dimensional vector $[0100\ldots001]$. Then, the eigenvalues are given by $\lambda_l(\boldsymbol{A}_0) = 2\cos(2\pi l/n_1)$ and applying the formula in (37) the eigenvalues of $\boldsymbol{A}$ are $\lambda_{l,m}(\boldsymbol{A}) = 2\cos(2\pi l/n_1) + 2\cos(2\pi m/n_1)$ for $l, m = 0, \ldots, n_1 - 1$. The second largest eigenvalue of $\boldsymbol{A}$ is obtained for $l = 1$ and $m = 0$ and is given by $\lambda_{1,0}(\boldsymbol{A}) = 2\cos(2\pi/n_1) + 2$. Thus the eigenvalues of the Laplacian matrix can be expressed as $\lambda_i(\boldsymbol{L}) = 4 - \lambda_i(\boldsymbol{A})$ and the algebraic connectivity of the network is given by $\lambda_2(\boldsymbol{L}) = 2 - 2\cos(2\pi/n_1)$.

REFERENCES

[1] R. O. Saber and R. M. Murray, "Agreement problems in networks with directed graphs and switching topology," in *Proc. 42nd IEEE Conf. on Decision Contr. (CDC 03)*, Maui, HI, Dec. 2003.

[2] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, pp. 215–233, Jan. 2007.

[3] S. Barbarossa and G. Scutari, "Decentralized maximum-likelihood estimation for sensor networks composed of nonlinearly coupled dynamical systems," *IEEE Trans. Signal Process.*, vol. 55, pp. 3456–3470, Jul. 2007.

[4] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 52, pp. 2508–2530, Jun. 2006.

[5] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Math. J.*, vol. 23, pp. 298–305, 1973.

[6] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Syst. Contr. Lett.*, vol. 53, pp. 65–78, Sept. 2004.

[7] S. Boyd, "Convex optimization of graph Laplacian eigenvalues," in *Proc. Int. Congr. Math.*, Madrid, Spain, 2006, pp. 1311–1319.

[8] R. Olfati-Saber, "Ultrafast consensus in small-world networks," in *Proc. 2005 Amer. Control Conf. (ACC 05)*, Portland, OR, Jun. 2005, pp. 2371–2378.

[9] S. Kar, S. Aldosari, and J. M. F. Moura, "Topology for distributed inference on graphs," *IEEE Trans. Signal Process.*, vol. 56, pp. 2609–2613, Jun. 2008.

[10] A. Ghosh and S. Boyd, "Growing well-connected graphs," in *Proc. 45nd IEEE Conf. on Decision Control (CDC 06)*, San Diego, CA, Dec. 2006, pp. 6605–6611.

[11] A. D. G. Dimakis, A. D. Sarwate, and M. J. Wainwright, "Geographic gossip: Efficient averaging for sensor networks," *IEEE Trans. Signal Process.*, vol. 56, pp. 1206–1216, Mar. 2008.

[12] A. K. Das and M. Mesbahi, "K-node connectivity power efficient topologies in wireless networks: A semidefinite programming approach," in *Proc. IEEE Global Telecomm. Conf. (GLOBECOM)*, St. Louis, MO, Nov.–Dec. 2005, vol. 1, pp. 468–473.

[13] R. Ramanathan and R. Rasales-Hain, "Topology control of multihop wireless networks using transmit power adjustement," in *Proc. Ann. Joint Conf. IEEE Comput. Commun. Soc. (INFOCOM)*, Tel Aviv, Israel, Mar. 2000, vol. 2, pp. 404–413.

[14] A. Giridhar and P. R. Kumar, "Toward a theory of in-network computation in wireless sensor networks," *IEEE Commun. Mag.*, vol. 44, pp. 98–107, Apr. 2006.

[15] S. Barbarossa, G. Scutari, and A. Swami, "Achieving consensus in self-organizing wireless sensor networks: The impact of network topology on energy consumption," in *Proc. IEEE ICASSP 2007*, Honolulu, HI, Apr. 2007.

[16] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York: Springer-Verlag, 2001, vol. 207, Graduate Texts in Mathematics.

[17] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[18] A. Jeflea, "A parametric study for solving nonlinear fractional problems," *Ann. St. Univ. Ovidius Constanta*, vol. 11, pp. 87–92, 2003.

[19] W. Dinkelbach, "On nonlinear fractional programming," *Manage. Sci.*, vol. 13, pp. 492–498, Mar. 1967.

[20] R. Jagannathan, "On some properties of programming problems in parametric form pertaining to fractional programming," *Manage. Sci.*, vol. 12, pp. 609–615, Mar. 1966.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York: Cambridge Univ. Press, 2004.

[22] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Trans. Autom. Control*, vol. 50, pp. 1867–1872, Nov. 2005.

[23] T. C. Aysal and K. E. Barner, "Convergence of consensus models with stochastic disturbances," *IEEE Trans. Inf. Theory*, vol. 56, pp. 4101–4113, Aug. 2010.

[24] C. Bordenave, "Eigenvalues of Euclidean Random Matrices," in *Random Structures & Algorithms*. New York: Wiley, Dec. 2008, vol. 33, pp. 515–532.

[25] S. Rai, "The spectrum of a random geometric graph is concentrated," *J. Theoret. Probabil.*, vol. 20, pp. 119–132, Jun. 2007.

[26] M. Mezard, G. Parisi, and A. Zee, "Spectra of euclidean random matrices," *Nucl. Phys. B*, vol. 559, pp. 689–701, Oct. 1999.

[27] P. Gupta and P. Kumar, "Critical Power for Asymptotic Connectivity in Wireless Networks," in *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming, W. M. McEneaney, G. Yin, and Q. Zhang (Eds.)*. Boston, MA: Birkhauser, 1998, pp. 547–566.

[28] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Mixing times for random walks on geometric graphs," in *Proc. Workshop on Analytic Algorithmics Combinator. (ANALCO'05)*, Vancouver, BC, Canada, Jan. 2005.

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